A DOUBLE-PROJECTION ALGORITHM FOR EQUILIBRIUM PROBLEMS

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RESUMO

Propomos um algoritmo explícito de duas fases para resolver Problemas de Equilíbrio em Espaços Euclidianos. A idéia é executar, a cada iteração, reflexões relativas a hiperplanos e uma projeção sobre um semi-espaço. A convergência da sequência gerada é provada sob hipóteses existentes na literatura. Experimentos numéricos são relatados.

PALAVRAS CHAVE. Problema de Equilíbrio, Operador de Reflexão, Métodos de Projeção, Programação Matemática.

ABSTRACT

We propose an explicit two-phase algorithm for solving Equilibrium problems in Euclidean spaces. The idea is perform, at each iteration, reflections related to hyperplanes and a projection onto a halfspace. The convergence of the generated sequence is proved under assumptions considered in the literature. Numerical experiments are reported.

KEYWORDS. Equilibrium problem, Reflection operator, projection methods, Mathematical programming.

1. Introduction

Let $\mathbb{R}^n$ be an Euclidean space with inner product $\langle\cdot,\cdot\rangle$ and define $\|x\| := \langle x, x\rangle^{\frac{1}{2}}$, for all $x \in \mathbb{R}^n$. Let C be a nonempty closed convex subset of $\mathbb{R}^n$ and $f\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ a bifunction such that $f(x, x) = 0$ for all x. We consider the following *Equilibrium problem* $\mathrm{EP}(f, C)$:

$$(EP)\begin{cases} \text{Find } \ x^* \in \ C \ \text{ such that} \\ f(x^*, y) \geq 0 \quad \forall \ y \ \in C. \end{cases} \tag{1}$$

The solution set of $\mathrm{EP}(f, C)$ is denoted by $S(f, C)$.
Formulation (1) provides a unified framework for several problems in the sense that it includes, as particular cases, optimization problems, Nash equilibria problems, complementarity problems, fixed point problems, variational inequalities and vector minimization problems; see, for example, Blum and Oettli (1994). Numerical algorithms for solving the equilibrium problem have been proposed based on the auxiliary problem principle, the proximal point technique and projections onto the original set or onto approximations; see for instance Iusem and Sosa (2010), Konnov (2003), Nguyen et al (2009), Quoc and Muu (2010), Lyashko et al (2011), Santos and Scheimberg (2011a), Santos and Scheimberg (2011b) and the references therein.

In this paper we assume the usual condition that the function, $f(x, \cdot)\colon \mathbb{R}^n \to \mathbb{R}$, is convex for all $x \in \mathbb{R}^n$ and we consider the case in which C is of the form:

$$C = \{x \in \mathbb{R}^n \, : \, g(x) \leq 0\}, \tag{2}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a nonlinear convex function. Differentiability of g is not assumed and the representation (2) is therefore rather general, because any finite system of inequalities $g_j(x) \leq 0$ with $j \in J$, where all the g_j's are convex, may be represented as in (2) with $g(x) = \sup\{g_j(x) \, : \, j \in J\}$. This representation for the feasible set has been widely used in the literature; see Fukushima (1986), Censor and Gibali (2008), Bello Cruz and Iusem (2010b). Furthermore, it is well known that any closed convex set C can be represented by (2) for an appropriate convex function g.

Our objective is to develop an iterative algorithm for solving (1)-(2), such that each iteration consists, essentially, of two phases. In the first phase (inner loop), starting from an infeasible point x^{k-1}, a movement towards feasibility is performed by using reflections related to hyperplanes, obtaining a point z^k. In the second phase, an approximation to a solution of the problem is improved by considering a projected-type subgradient step, generating a point x^k. The projection is done onto a suitable half-space containing the solution set. Actually, the method generates a feasible sequence $\{z^k\}$, and a sequence $\{x^k\}$ containing infeasible points. It is an implementable method with low computational cost since only closed formulae are calculated.

The inner loop was given in Konnov (1998) embedded in a projection method for variational inequalities. We prove that the whole sequences generated by the algorithm are convergent to a solution of the problem, under the standard assumptions of pseudomonotonicity of the bifunction, upper semicontinuity of the function $f(\cdot, y)$, the boundedness of the subgradient of the function $f(x, \cdot)$ at x on bounded sets, existence of solutions and satisfying a condition of weak paramonotonicity type.

The rest of this paper is organized as follows. In Section 2 we present some theoretical results needed in our analysis. In Section 3 we state our algorithm formally. In Subsection 3.1 we present an inner loop, called Algorithm IA which finishes after a finite number of iterations. In Subsection 3.2 we introduce the global algorithm and in Subsection 3.3 we establish the convergence properties of the algorithm. Finally, in Section 4, we illustrate the behavior of the method by a numerical experiments.

2. Preliminary results

In this section, we present some definitions and results, needed in the convergence analysis of the proposed method.
First, we state two well known properties of orthogonal projections. We recall that, given a nonempty closed and convex subset C of $\mathbb{R}^n$, the orthogonal projection of $x \in \mathbb{R}^n$ onto C, denoted by $P_C(x)$, is the unique point in C, such that $\|P_C(x) - y\| \leq \|x - y\|$ for all $y \in C$.

Lemma 1 (Lemma 1, Solodov and Svaiter (2000)) *Let C be a nonempty closed and convex set in $\mathbb{R}^n$. For all $x, y \in \mathbb{R}^n$ and all $z \in C$, the following properties hold:*

i) $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|(P_C(x) - x) - (P_C(y) - y)\|^2$.

ii) $\langle x - P_C(x), z - P_C(x)\rangle \leq 0$.

Lemma 2 (Proposition 3.1, Qu and Xiu (2008)) *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Given $x \in \mathbb{R}^n$ and $v \in \partial g(x)$, let $C(x,v) := \{z \in \mathbb{R}^n : g(x) + \langle v, z - x\rangle \leq 0\}$. Then for any $y \in \mathbb{R}^n$:*

$$P_{C(x,v)}(y) = \begin{cases} y - \dfrac{g(x) + \langle v, y - x\rangle}{\|v\|^2} v & \text{if} \quad y \notin C(x,v) \\ y & \text{if} \quad y \in C(x,v). \end{cases}$$

Observe that when C is given by (2), $C \subseteq C(x,v)$ for all $x \in \mathbb{R}^n$.
Let us recall that, the projection of a point $y \in \mathbb{R}^n$ onto the hyperplane $H(x,v) = \{z \in \mathbb{R}^n : g(x) + \langle v, z - x\rangle = 0\}$ is given by $P_{H(x,v)}(y) = y - \dfrac{1}{\|v\|^2}[g(x) + \langle v, y - x\rangle]v$. It follows

$$P_{H(x,v)}(x) = x - \frac{1}{\|v\|^2} g(x)v. \tag{3}$$

For $S \subseteq \mathbb{R}^n$, the distance function $\operatorname{dist}(\cdot, S)$ is defined by $\operatorname{dist}(x,S) := \inf_{z\in S}\|z - x\|$. If S is a closed and convex set then $\operatorname{dist}(x,S) = \min_{z\in S}\|z - x\| = \|P_S(x) - x\|$.
Next, we establish four technical and elementary results to be used in the convergence analysis.

Lemma 3 *Let $\{\alpha_k\}$, $\{\beta_k\}$ be sequences of real numbers satisfying $\{\beta_k\} \subset [0, +\infty)$ and $\sum_{k=0}^{\infty}\beta_k = +\infty$. Suppose that $\displaystyle\sum_{k=0}^{\infty}\alpha_k\beta_k < +\infty$, then, $\liminf_{k\to+\infty}\alpha_k \leq 0$.*

Lemma 4 *Let $\{\nu_k\}$ and $\{\delta_k\}$ be nonnegative sequences of real numbers satisfying $\nu_{k+1} \leq \nu_k + \delta_k$ with $\sum_{k=1}^{+\infty}\delta_k < +\infty$. Then the sequence $\{\nu_k\}$ is convergent.*

From now on, we consider a bifunction $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $f(x,x) = 0$ for all $x \in \mathbb{R}^n$ and $f(x,\cdot)\colon \mathbb{R}^n \to \mathbb{R}$ is convex. The following subdifferential notion for bifunctions studied in Iusem (2011) will be a useful tool in our development.

Definition 1 *The diagonal subdifferential $\partial_2 f\colon \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ of a bifunction f at $x \in \mathbb{R}^n$, is given by*

$$\begin{aligned} \partial_2 f(x,x) &:= \{u \in \mathbb{R}^n\colon f(x,y) \geq \langle u, y - x\rangle + f(x,x) \;\; \forall \;\; y \in \mathbb{R}^n\} \\ &= \{u \in \mathbb{R}^n\colon f(x,y) \geq \langle u, y - x\rangle \;\; \forall \;\; y \in \mathbb{R}^n\}. \end{aligned} \tag{4}$$

The elements of the diagonal subdifferential are called diagonal subgradients; see, for example, Santos and Scheimberg (2011b) and the references therein.

3. An explicit reflection-projection algorithm

In this section, we examine an inner procedure for finding a feasible point by considering reflections related to hyperplanes. Also, we introduce an algorithm for solving EP(f, C) which uses the inner procedure and projections onto half-spaces.
From now on, we assume the following assumption.

(A1) There is a Slater point of g, i.e. a point $w \in \mathbb{R}^n$ such that $g(w) < 0$.

We consider that an oracle is available, which for any given $x \in \mathbb{R}^n$ computes the subgradient for the function g (or a diagonal subgradient of f), i.e., some $v \in \partial g(x)$ (or $u \in \partial_2 f(x, x)$). As usual in nonsmooth optimization Karas et al (2009), we do not assume that there is any control over which particular subgradients are computed by the oracle.

Remark 1 *Condition* (A1) *is a standard constraint qualification, see for instance Konnov (2003) for Equilibrium problems and Bello Cruz and Iusem (2010b), Karas et al (2009), Nedić and Ozdaglar (2009) for related problems.*

Algorithm IA
Data: $x \in \mathbb{R}^n$.
Output: $y \in C$.
Step 0: If $g(x) \leq 0$ set $y = x$, stop. Otherwise, set $y^0 = x$, $j := 0$.
Step 1: take $s^j \in \partial g(y^j)$. Calculate

$$y^{j+1} := y^j - \frac{2}{\|s^j\|^2} g(y^j) s^j, \quad j := j + 1. \tag{5}$$

Step 2: If

$$g(y^j) \leq 0 \qquad \text{take } \; j(y) = j, \;\; y = y^j, \text{ stop.} \tag{6}$$

Otherwise, go back to step 1.

Observe that $d(y^j, H_j) = d(y^{j+1}, H_j)$ where $d(y, H_j) = min_{z \in H_j} \; \|y - z\|$ and the hyperplane $H_j = \{z \in \mathbb{R}^n : g(y^j) + < s^j, z - y^j >\} = 0$ separetes y^j from $S(f, C)$.
We summarize the properties of Algorithm IA, which has $j(y)$ and $y^{j(y)}$ as output.

Proposition 1 *Let* $\{y^j\}$ *be defined by the inner algorithm where the initial point* y^0 *verifies* $g(y^0) > 0$. *Then,*

i) $\|y^{j+1} - y\| \leq \|y^j - y\| \;\; \forall \;\; y \in C$.

ii) The algorithm generates a finite number of iterations.

Proof:

i) See Lemma 9 in Konnov (1998).

ii) See Lemma 10 in Konnov (1998).

□

3.2. The Algorithm

Take a positive parameter ρ and real sequences $\{\rho_k\}$, $\{\lambda_k\}$ and $\{\beta_k\}$ verifying the following conditions:

$$\rho_k > \rho, \quad \beta_k > 0, \quad \lambda_k \in (0,1] \quad \forall\, k \in \mathbb{N}. \tag{7}$$

$$\sum_{k=0}^{\infty} \frac{\lambda_k\ \beta_k}{\rho_k} = +\infty, \qquad \sum_{k=0}^{\infty} \beta_k^2 < +\infty. \tag{8}$$

Algorithm DPA

Step 0: Choose $x^0 \in \mathbb{R}^n$. Set $k := 0$.

Step 1: Let $x^k \in \mathbb{R}^n$. If $g(x^k) \le 0$ then $z^k := x^k$
Otherwise apply Algorithm IA with $x = x^k$. Set $j(k) = j(y)$, $z^k = y^{j(k)}$.

Step 2: Given z^k, take $v^k \in \partial g(z^k)$, $u^k \in \partial_2 f(z^k, z^k)$ and let

$$C_k := C(z^k, v^k) = \{y \in \mathbb{R}^n : g(z^k) + \langle v^k, y - z^k\rangle \le 0\}.$$

Calculate $\eta_k := \max\{\rho_k, \|u^k\|\}$, and

$$x^{k+1} := (1-\lambda_k)z^k + \lambda_k P_{C_k}\left(z^k - \frac{\beta_k}{\eta_k}u^k\right). \tag{9}$$

Step 3: If $x^{k+1} = z^k$, stop. Otherwise, $k := k+1$ and go back to Step 1.

Next we show that the cost of the projection in (9) is negligible.

Lemma 5 *Assume that condition* (A1) *is satisfied. Then the iterate* x^{k+1} *has the following explicit formulae*

$$\begin{aligned} x^{k+1} &= (1-\lambda_k)z^k + \lambda_k P_{C_k}\left(z^k - \frac{\beta_k}{\eta_k}u^k\right) \\ &= z^k - \lambda_k\left(\frac{\beta_k}{\eta_k}u^k + \frac{1}{\|v^k\|^2}\max\left\{0, g(z^k) - \frac{\beta_k}{\eta_k}\langle u^k, v^k\rangle\right\} v^k\right). \end{aligned}$$

Proof: Follows directly from Lemma 2. □

3.3. Convergence analysis of Algorithm DPA

We begin the convergence analysis of the algorithm by studying the case where the sequence is finite.

Proposition 2 *Assume that* (A1) *holds. Let* $\{x^k\}$ *and* $\{z^k\}$ *be the sequences generated by Algorithm DPA. If* $x^{k+1} = z^k$*, then,* x^{k+1} *is a solution of* $EP(f, C)$*.*

Proof: Let x^{k+1} be the last term of $\{x^k\}$. Suppose that $x^{k+1} = z^k$. From (9) and $\lambda_k > 0$, we get that

$$z^k = P_{C_k}\left(z^k - \frac{\beta_k}{\eta_k}u^k\right).$$

Thus, by Lemma 1(ii), we have

$$z^k \in C_k \ \text{ and } \ \frac{\beta_k}{\eta_k}\left\langle u^k, z - z^k\right\rangle \ge 0 \quad \forall z \in C_k. \tag{10}$$

Since $z^k \in C \subseteq C_k$ and $\dfrac{\beta_k}{\eta_k} > 0$ for all k, by (10), we get

$$\langle u^k, z - z^k \rangle \geq 0 \quad \forall\, z \in C.$$

Therefore, from $u^k \in \partial_2 f(z^k, z^k)$, and (4) we deduce that

$$f(z^k, z) \geq \langle u^k, z - z^k \rangle \geq 0 \quad \forall\, z \in C.$$

Hence, we conclude that $x^{k+1} = z^k \in S(f, C)$. □

From now on we assume that the sequences $\{x^k\}$ and $\{z^k\}$ generated by Algorithm DPA infinite.
We present two results that are needed for the convergence analysis of Algorithm DPA.

Proposition 3 *Assume that (A1) is satisfied. Then, it holds:*

i) $\lim_{k\to+\infty} \|x^{k+1} - z^k\| = 0,$

ii) $\lim_{k\to+\infty} \operatorname{dist}(x^k, C) = 0,$

iii) all weak cluster points of $\{x^k\}$ *belong to* C*.*

Proof:

i) It follows from (7), (9) and (6) that

$$\begin{aligned} \|x^{k+1} - z^k\|^2 &= \lambda_k \|P_{C_k}(z^k - \tfrac{\beta_k}{\eta_k} u^k) - z^k\|^2 \\ &= \|P_{C_k}(z^k - \tfrac{\beta_k}{\eta_k} u^k) - P_{C_k}(z^k)\|^2. \end{aligned}$$

Therefore, combining the above inequality and Lemma 1, it results

$$\begin{aligned} \|x^{k+1} - z^k\|^2 &\leq \beta_k \eta_k^{-1} \left\|u^k\right\| \\ &\leq \beta_k. \end{aligned}$$

The conclusion now follows from (i) and (8).

ii) This result follows from (i).

iii) This conclusion is an immediate consequence of (ii).

□

Lemma 6 *Assume that* (A1) *is satisfied. Then, for any* $x \in C$ *and for all* $k \in \mathbb{N}$*, it holds:*

i) $\|z^k - x\| \leq \|x^k - x\|.$

ii) $\|x^{k+1} - x\|^2 \leq \|z^k - x\|^2 + \beta_k^2 + \dfrac{2\lambda_k \beta_k}{\eta_k} \langle u^k, x - z^k \rangle.$

Proof: Let $x \in C$ and $k \in \mathbb{N}$.

i) If $g(x^k) \leq 0$ then the conclusion is trivial.

Otherwise, from Proposition 1(ii), we have

$$\begin{aligned} \|z^k - x\| &= \|y^{j(k)} - x\| \\ &\leq \|y^{j(k)-1} - x\| \leq \ldots \leq \|y^0 - x\| \\ &= \|x^k - x\|. \end{aligned}$$

ii) Recalling that $0 < \lambda_k \leq 1$, $\eta_k = \max\{\rho_k, \|u^k\|\}$ and $\rho_k > \rho > 0$, we have that

$$\frac{\|u^k\|}{\eta_k} \leq 1 \text{ and } \frac{\lambda_k}{\eta_k} \leq \frac{1}{\rho}. \tag{11}$$

By Lemma 1(i), the above inequalities and taking $x = (1-\lambda_k)x + \lambda_k P_{C_k}(x)$, we get

$$\begin{aligned} \|x^{k+1} - x\|^2 &\leq (1-\lambda_k)\|z^k - x\|^2 + \lambda_k \|P_{C_k}(z^k - \tfrac{\beta_k}{\eta_k}u^k) - P_{C_k}(x)\|^2 \\ &\leq (1-\lambda_k)\|z^k - x\|^2 + \lambda_k \left\|z^k - \tfrac{\beta_k}{\eta_k}u^k - x\right\|^2 \\ &= (1-\lambda_k)\|z^k - x\|^2 + \lambda_k\|z^k - x\|^2 + \lambda_k \frac{\|u^k\|^2}{\eta_k^2}\beta_k^2 + \frac{2\lambda_k\beta_k}{\eta_k}\langle u^k, x - z^k\rangle \\ &\leq \|z^k - x\|^2 + \beta_k^2 + \frac{2\lambda_k\beta_k}{\eta_k}\langle u^k, x - z^k\rangle. \end{aligned}$$

□

The following assumptions allows us to establish the boundedness of the sequences, $\{z^k\}$ and $\{x^k\}$, generated by Algorithm DPA.

(A2) The solution set $S(f, C)$ is nonempty.

(A3) f is a pseudomonotone bifunction.

Proposition 4 *Assume that* (A1)-(A3) *are satisfied. Then,*

(i) the sequences $\{\|x^k - \overline{x}\|\}$ *and* $\{\|z^k - \overline{x}\|\}$ *are convergent, for all* $\overline{x} \in S(f, C)$,

(ii) the sequences $\{\|x^k\|\}$ *and* $\{\|z^k\|\}$ *are bounded.*

Proof:

i) Let $\overline{x} \in S(f, C)$. By Lemma 6, we have that

$$\begin{aligned} \|z^{k+1} - \overline{x}\|^2 &\leq \|x^{k+1} - \overline{x}\|^2 \\ &\leq \|z^k - \overline{x}\|^2 + \beta_k^2 + 2\frac{\lambda_k\beta_k}{\eta_k}\langle u^k, \overline{x} - z^k\rangle \\ &\leq \|x^k - \overline{x}\|^2 + \beta_k^2 + 2\frac{\lambda_k\beta_k}{\eta_k}\langle u^k, \overline{x} - z^k\rangle. \end{aligned} \tag{12}$$

Since $\bar{x} \in S(f, C)$, we have $f(\bar{x}, z^k) \geq 0$. Therefore, by (A3) and the definition of $u^k \in \partial_2 f(z^k, z^k)$ we deduce that

$$\begin{aligned} \langle u^k, \bar{x} - z^k\rangle &\leq f(z^k, \bar{x}) \\ &\leq 0. \end{aligned}$$

By the above inequality and (12), we have that

$$\|z^{k+1} - \overline{x}\|^2 \leq \|z^k - \overline{x}\|^2 + \beta_k^2,$$

which together with Lemma 6(i), it results

$$\|x^{k+1} - \overline{x}\|^2 \leq \|x^k - \overline{x}\|^2 + \beta_k^2.$$

Our conclusion follows from Lemma 4.

ii) This result follows from (i). □

We establish the convergence of the whole sequence $\{z^k\}$ to a solution of the equilibrium problem under the following conditions.

(A4) $\partial_2 f$ is bounded on bounded sets.

(A5) $f(\cdot, z)$ is an upper semicontinuous for every $z \in \mathbb{R}^n$.

(A6) Let $x^* \in S(f,C)$ and $\bar{x} \in C$. If $f(\bar{x}, x^*) = f(x^*, \bar{x}) = 0$ then $\bar{x} \in S(f,C)$.

Proposition 5 *Assume that* (A1)-(A4) *hold. Then,*

$$\limsup_{k\to+\infty} f(z^k, \bar{x}) = 0 \quad \forall \;\; \bar{x} \in S(f,C).$$

Proof: Let $\bar{x} \in S(f,C)$. Then, by Lemma 6, we have

$$\|z^{k+1} - x\|^2 \le \|z^k - x\|^2 + \beta_k^2 + \frac{2\lambda_k\beta_k}{\eta_k} f(z^k, \bar{x}), \tag{13}$$

that is,

$$\frac{2\lambda_k\beta_k}{\eta_k}[-f(z^k,\bar{x})] \le \|z^k - x\|^2 - \|z^{k+1} - x\|^2 + \beta_k^2.$$

Therefore, it results that

$$\begin{aligned} 2\sum_{k=0}^{m} \frac{\lambda_k\beta_k}{\eta_k}[-f(z^k,\bar{x})] &\le \|z^0 - x\|^2 - \|z^{m+1} - x\|^2 + \sum_{k=0}^{m}\beta_k^2 \\ &\le \|z^0 - x\|^2 + \sum_{k=0}^{m}\beta_k^2. \end{aligned} \tag{14}$$

Hence, by using (14) we obtain

$$\sum_{k=0}^{+\infty} \frac{\lambda_k\beta_k}{\eta_k}[-f(x^k,\bar{x})] < +\infty. \tag{15}$$

On the other hand, (A4) and Proposition 4 imply that $\{\|u^k\|\}$ is bounded.

Therefore, there exists $L \ge \rho$ such that $\left\|u^k\right\| \le L$ for all $k \in \mathbb{N}$, so it results

$$\frac{\eta_k}{\rho_k} = \max\{1, \rho_k^{-1}\|u^k\|\} \le \frac{L}{\rho} \quad \forall \;\; k \;\; \in \mathbb{N}.$$

Consequently, we have

$$\frac{\lambda_k\beta_k}{\eta_k} \ge \frac{\rho}{L}\frac{\lambda_k\beta_k}{\rho_k},$$

which together with (8) implies that

$$\sum_{k=0}^{\infty} \frac{\lambda_k\beta_k}{\eta_k} = +\infty.$$

Using the above inequalities and Lemma 3, we conclude that

$$\liminf_{k\to+\infty}[-f(z^k,\bar{x})] \le 0,$$

that is,

$$\limsup_{k\to+\infty} f(z^k,\bar{x}) \ge 0.$$

□

Theorem 1 *Assume that* (A1)-(A6) *hold. Then, the whole sequence* $\{x^k\}$ *converges to a solution of the equilibrium problem.*

Proof: Let $x^* \in S(f, C)$, by Proposition 5 there exists a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that

$$\limsup_{k\to+\infty} f(x^k, x^*) = \lim_{j\to+\infty} f(x^{k_j}, x^*). \tag{16}$$

In view of Proposition 4, $\{x^{k_j}\}$ is bounded. So, there is $\bar{x} \in C$ and a subsequence of $\{x^{k_j}\}$, without loss of generality, namely $\{x^{k_j}\}$, such that

$$\lim_{j\to+\infty} x^{k_j} = \bar{x}. \tag{17}$$

Combining assumption (A5) together with Proposition 5 it follows

$$\begin{array}{rcl} f(\overline{x}, x^*) & \geq & \limsup_{j\to+\infty} f(x^{k_j}, x^*) \\ & = & \lim_{j\to+\infty} f(x^{k_j}, x^*) \\ & = & \limsup_{k\to+\infty} f(x^k, x^*) \\ & = & 0. \end{array} \tag{18}$$

From assumption (A3) we have $f(\overline{x}, x^*) \leq 0$, so, it results

$$f(\overline{x}, x^*) = 0. \tag{19}$$

Therefore, (A6) implies that $\overline{x} \in S(f, C)$.

Using again Proposition 4 we obtain that the sequence $\{\|x^k - \bar{x}\|\}$ is convergent, which together with (17) it yields

$$\lim_{k\to+\infty} x^k = \bar{x}, \quad \bar{x} \in S(f, C).$$

$\square$

Remark 2 *Condition* (A4) *has been considered in Santos and Scheimberg (2011b), Iusem and Sosa (2003) for equilibrium problems, in Bello Cruz and Iusem (2010a) for variational inequalities, and in Alber et al (1998), Polyak (1969) for optimization problems. This condition is satisfied, for exemple, if* f *is a monotone bifunction.*

We illustrate all the assumptions by the following example.

Example 1 *We consider the equilibrium problem defined by* $C = [-1, 1]$ *and* $f(x, y) = |x|(y - x)$. *Let us observe that* $S(f, C) = \{-1, 0\}$ *is solution set of* $EP(f, C)$ *and* (A1)-(A5) *hold.*

4. Numerical results

In this section we illustrate the behavior of the algorithm DPA by considering two numerical tests coded in SCILAB 5.3.2 on a 2GB RAM Intel Atom N450.

Example 2 *Consider a general equilibrium problem given in Bao et al (2005) and defined by* $C = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : g(x) = \max_{i=1,2,3,4}\{g_i(x)\} \leq 0\}$, *with* $g_1(x) = x_1^2 - x_2 - 1$, $g_2(x) = x_3^2 - x_4 - 1$, $g_3(x) = 2x_1 + x_2 - 3$, $g_4(x) = 2x_3 + x_4 - 3$ *and* $f(x, y) = \sum_{j=1}^4 \phi_j(x)(y_j - x_j)$ *where*

$$\begin{array}{rcl} \phi_1(x) & = & x_1 - 2x_2 \\ \phi_2(x) & = & -2x_1 + 4x_2 \\ \phi_3(x) & = & x_3 - 2x_4 \\ \phi_4(x) & = & -2x_3 + 4x_4 \end{array}$$

The solution point for this problem is $\overline{x} = (1.2, 0.6, 1.2, 0.6)$. *Let us observe that the derivative of* $f(x, \cdot)$ *with respect to* y, *at* x, *is* $\partial_2 f(x, x) = (\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x))$.

The algorithms start from $x^0 = (100, 100, 100, 100)$. For Algorithm DPA, we set $\lambda_k = \frac{k}{k+1}$ and $\beta_k = \frac{72}{10k}$. For the Extragradient Algorithms, we consider the same Bregman function, given by $G(x) = \frac{1}{2}\|x\|^2$, like in Tran et al (2008). We take $\rho = 0.2$ for EA1 and $\rho = 0.6$ for EA2, $\alpha = \frac{\rho}{10}$, $\theta = 0.9$, which are the best choice of the parameters $\rho \in \{0.1, 0.2, \ldots, 2\}$ and $\theta \in \{0.1, 0.2, \ldots, 0.9\}$. For IPSM, we take $\beta_k = \frac{73}{10k}$ and $\gamma_k = \max\{3.3, \|(\phi_1(x^k), \phi_2(x^k), \phi_3(x^k), \phi_4(x^k))\|\}$. Table 1 shows the performance of the algorithms.

Table 1: Comparison of DPA with others algorithms

Algorithm	Iterations	Inner loops	cpu(s)
DPA	10	36	0.318
IPSM	13	2	1.956
EA1	2	4	4.526
EA2	8	46	11.602

Note that in terms of cpu time Algorithm DPA takes advantage on the others algorithms.

Example 3 *We Consider the Rosen-Suzuki optimization problem taken from Problem 43 of Hock and Schittkowski (1981) and its reformulation as an equilibrium problem. The bifunction is given by* $f(x, y) = \phi(y) - \phi(x)$ *with*

$$\phi(x) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4.$$

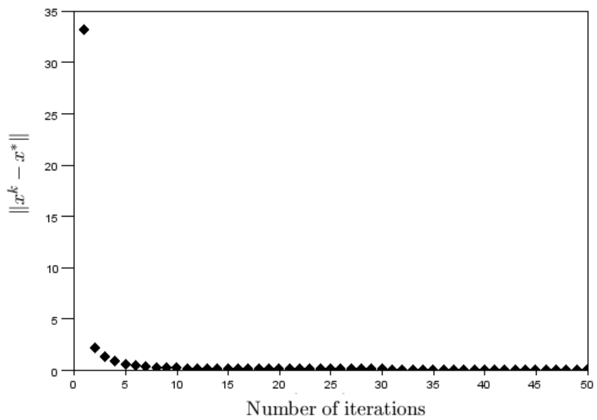
Hence, $f_y(x, x) = (2x_1 - 5, 2x_2 - 5, 4x_3 - 21, 2x_4 + 7)^T$.
The constraint set is defined by $C = \{x \in \mathbb{R}^4 : \ g_i(x) \leq 0, \ i = 1, 2, 3\}$, *where*

$$\begin{aligned} g_1(x) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8, \\ g_2(x) &= x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10, \\ g_3(x) &= 2x_1^2 + x_2^2 + x_3^2 + 2x_1 - x_2 - x_4 - 5. \end{aligned}$$

The optimal point is $\overline{x} = (0, 1, 2, -1)$.

Figure 1 gives the evolution of the term $\|x^k - x^*\|$ along of 50 iterations, by considering the median of five randomly generated starting points $x^0 \in \mathbb{R}^4$. We set $\lambda_k = \frac{k}{k+1}$ and $\beta_k = \frac{347}{100k}$.

Figure 1: Evolution of $\|x^k - x^*\|$ along of 50 iterations



5. Conclusion

We analyzed an explicit two-phase algorithm for solving Equilibrium problems in Euclidean spaces. In the first phase, a reflection step is performed. Secondly, a projection onto a halfspace is calculated. Hence, it is an implementable method with low computational cost since only closed formulae are computed. The convergence of the generated sequence is proved under mild assumptions. To illustrate the numerical behavior of the algorithm, two examples are reported.

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