ANALYSIS AND SIMULATION OF DISCRETE LQR CONTROL VIA LMIS APPLIED IN THE ALTERNATIVE BOOST CONVERTER

Abstract— This work proposes the application and analysis of a LQR (Linear-quadratic regulator) servo tracking control optimized via Linear Matrix Inequalities (LMIs) in a boost converter based on the three-state switching cell (3SSC). The aforementioned converter is an improved version of the conventional boost topology which presents high efficiency and is adequate for high-power high-current applications. The discrete LQR using LMI is a modified optimization of the continuous model. The proposed strategy is compared with LQR servo tracking. Robustness and time response analysis are also performed considering the LTV (linear time-varying) system.

Keywords— *Boost* Converter , Robust Control, LQR-LMI, LMI optimization, LTV systems.

1 Introduction

The development of research works involving robust control with LMI has increased significantly in the last dew years. Besides, applications using LMI optimization have been proposed as novel solutions for problems involving robust control

Besides, the application of robust control applied to static power converters has been the focus of many studies because they represent nonlinear time-varying systems where numerous parameters are involved and must be properly controlled. The works proposed by Olalla et al. (2012) and Olalla et al. (2009) are examples of robust control using optimization applied to dc-dc converters. The work developed by Yao et al. (2011) applies LMI constraints to a boost converter operating in continuous conduction mode (CCM). Moreover, the importance of good tracking response in power converters has also been studied. El Beid and Doubabi (2014) apply the servo tracking topology using fuzzy logic to a conventional boost converter.

The strategy proposed in this paper is applied to a 3SSC-based boost converter (Bascopé and Barbi, 2000). The discrete LQR via LMI strategy is also compared with a discrete LQR.

The LQR-LMI control strategy was proposed by Ghaoui et al. (1992) and was applied to continuous time systems. Inspired in such model optimization, this paper proposes the LQR-LMI based on discrete time. Besides, the analysis is performed considering the LTV model operating according to the conditions stated by Kothare et al. (1996). Moreover, the servomechanism model designed by Levine (1999) and Fadali (2009) is applied to the discrete control of a boost converter. Finally, robustness analysis of the control strategies is performed by simulation.

2 Boost converter

The boost converter using the 3SSC was initially proposed by Bascopé and Barbi (2000). The magnetic elements are designed for twice the switching frequency, with consequent reduced size, weigh, and volume result without the increase of switching losses. High efficiency results, making this approach adequate for high power levels. The input voltage varies from 24 V to 33.6 V and must be stepped up to 48 V. The converter model was obtained according to the guidelines given by Orellana-Lafuente et al. (2010), Reis et al. (2011). Figure 1 shows the converter proposed by Bascopé and Barbi (2000) while Table 1 presents its respective parameters.

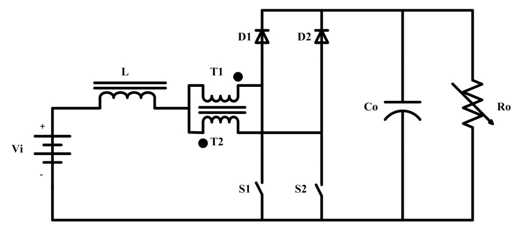

Figure 1: Boost converter using the 3SSC.

Table 1: Converter parameters.

Converter	
Parameters	**Values**
Input voltage (V_i or V_g)	24-33.6[V]
Output voltage (V_o)	48 [V]
D (rated duty cycle)	0,7
Switching frequency(f_s)	25[kHz]
Sample period(T_s)	1[ms]
Filter inductor (L)	70[μH]
Output filter capacitor (C_o)	2200[μF]
Equivalent series resistance (R_{se})	29[mΩ]
Load (R_o)	38.4-76.8[Ω]
Output power	30-60[W]

According to Middlebrook and Cuk (1976), the boost converter model in CCM can be obtained by using state-space averaging (SSA). For this purpose, Middlebrook and Cuk (1976) presented a methodology to obtain the average state-space model of a converter. By definition, the average state space combines the switching states S_1 and S_2, considering that the converter operates in steady state. Therefore, states S_1 and S_2 can be defined as :

- State S_1:

$$A_{1eq} = \begin{bmatrix} -\frac{R_{Leq}}{L_{eq}} & 0 \\ 0 & -\frac{1}{(R_{oeq}+R_{ceq})C_{eq}} \end{bmatrix},$$
$$B_{1eq} = \begin{bmatrix} \frac{1}{L_{eq}} \\ 0 \end{bmatrix},$$
$$C_{1eq} = \begin{bmatrix} 0 & \frac{R_{oeq}}{R_{oeq}+R_{ceq}} \end{bmatrix},$$
$$D_{1eq} = 0, \tag{1}$$

- State S_2:

$$A_{2eq} = \begin{bmatrix} \frac{R_{Leq}+R_{ceq}||\mathrm{R}_{eq}}{L_{eq}} & -\frac{\mathrm{R}_{eq}}{L_{eq}(\mathrm{R}_{oeq}+R_{ceq})} \\ -\frac{\mathrm{R}_{oeq}}{C_{eq}(\mathrm{R}_{oeq}+R_{ceq})} & -\frac{1}{C_{eq}(\mathrm{R}_{eq}+R_{ceq})} \end{bmatrix},$$
$$B_{2eq} = \begin{bmatrix} \frac{1}{L_{eq}} \\ 0 \end{bmatrix},$$
$$C_{2eq} = \begin{bmatrix} \mathrm{R}_{oeq}||R_{ceq} & \frac{\mathrm{R}_{oeq}}{\mathrm{R}_{oeq}+R_{ceq}} \end{bmatrix},$$
$$D_{2eq} = 0, \tag{2}$$

The aforementioned states are used in the composition of the system model, resulting in the following expressions:

$$\begin{aligned} \dot{\chi} &= A\chi + \left(\left(A_{1eq} - A_{2eq} \right) X + \left(B_{1eq} - B_{2eq} \right) V_g \right) d, \\ v_o &= C\chi + \left(\left(C_{1eq} - C_{2eq} \right) X + \left(D_{1eq} - D_{2eq} \right) V_g \right) d, \end{aligned} \tag{3}$$

The state space can be defined as:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{4}$$

by comparing the last expressions, it gives:

$$A = A_{1eq}D_{eq} + A_{2eq}(1 - D_{eq}), \tag{5}$$

$$B = \left(\left(A_{1eq} - A_{2eq} \right) X + \left(B_{1eq} - B_{2eq} \right) V_g \right), \tag{6}$$

$$C = C_{1eq}D_{eq} + C_{2eq}(1 - D_{eq}), \tag{7}$$

$$D = \left(\left(C_{1eq} - C_{2eq} \right) X \right), \tag{8}$$

where D_{eq} is the equivalent duty cycle, $\mathcal{X}$ is the state variable, and d is duty cycle perturbation. According to (4), parameters $\mathcal{X}$ and d can be associated to x and u, repectively. Therefore, the following expressions are valid:

$$X = \frac{V_{geq}}{R'} \begin{bmatrix} 1 \\ \left(1 - D_{eq} \right) R_{oeq} \end{bmatrix}, \tag{9}$$

$$V_{oeq} = Y = \frac{V_g \left(1 - D_{eq} \right) R_{oeq}}{R'}, \tag{10}$$

where $R' \triangleq (1 - D_d)^2 R_{oeq} + R_{Leq} + D_d (1 - D_d) (R_{ceq}||R_{oeq})$.

Expression (8) shows that $D \neq 0$. It is possible to state that this model is more accurate than that proposed by Montagner and Dupont (2010), because it considers the presence of the capacitor intrinsic series resistance and its influence on the output voltage.

2.1 Servomechanism model

Figure 2 shows the servomechanism proposed to control the power converter. This approach is similar to continuous model proposed by Levine (1999) and is in accordance with the discrete model introduced by Fadali (2009).

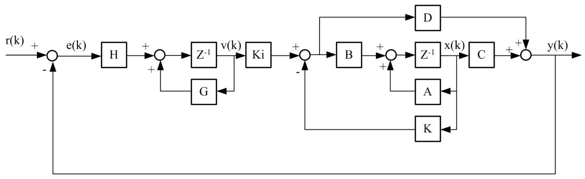

Figure 2: Proposed discrete servomechanism. .

The servo expressions are:

$$x\left(k+1\right) = Ax\left(k\right) + Bu\left(k\right), \tag{11}$$

$$v\left(k+1\right) = Gv\left(k\right) + He\left(k\right), \tag{12}$$

$$y\left(k\right) = Cx\left(k\right) + Du\left(k\right), \tag{13}$$

$$u\left(k\right) = -Kx\left(k\right) + Kiv\left(k\right), \tag{14}$$

$$e\left(k\right) = r\left(k\right) - y\left(k\right), \tag{15}$$

where A, B, C and D are the matrices that represent the state-space model and G, H are the matrices that correspond to the proposed servo control. According to Fadali (2009), the integrator is approached by forward Euler method, where the open loop matrices are given by:

$$\hat{A} = \begin{bmatrix} A & 0 \\ -HC & G \end{bmatrix} \quad e \quad \hat{B} = \begin{bmatrix} B \\ -HD \end{bmatrix}. \tag{16}$$

where $G = 1$ and $H = 1$. The closed-loop model of the servomechanism is given by:

$$\begin{bmatrix} x\left(k+1\right) \\ v\left(k+1\right) \end{bmatrix} = \begin{bmatrix} A - BK & BKi \\ -H\left(C - DK\right) & G - HDKi \end{bmatrix} \begin{bmatrix} x\left(k\right) \\ v\left(k\right) \end{bmatrix} + \\ + \begin{bmatrix} 0 \\ H \end{bmatrix} r\left(k\right), \tag{17}$$

$$y\left(k\right) = \begin{bmatrix} \left(C - DK\right) & DKi \end{bmatrix} \begin{bmatrix} x\left(k\right) \\ v\left(k\right) \end{bmatrix} \tag{18}$$

such that

$$A_{mf} = \begin{bmatrix} A - BK & BKi \\ -H\left(C - DK\right) & G - HDKi \end{bmatrix}, \; B_{mf} = \begin{bmatrix} 0 \\ H \end{bmatrix}, \\ C_{mf} = \begin{bmatrix} \left(C - DK\right) & DKi \end{bmatrix}, \; D_{mf} = 0. \tag{19}$$

where A_{mf}, B_{mf}, C_{mf} and D_{mf} are the closed-loop matrices.

2.2 Polytopic Analysis and Respective Uncertainties

The boost converter uncertainties are represented in Table 2. According to Aguirre (2008), polytopes are set polygon limits. Polytopes represent a convex skin with finite set of vertices, where all elements can be obtained by the convex combination of its respective vertices.

Table 2: Parametric Uncertainties.

Uncertainty Design	
Input voltage (Vg)	24-33.6V
Load(W)	30-60W

Table 2 states the polytopic uncertainty and constrains of the process, which are the input voltage $V_i \in [24.8, 33.6]$ and load variations $Pot \in [30, 60]$. These variables are included the polytopic format $p_i \in [\underline{p}_i, \bar{p}_i]$ (Olalla et al., 2009).

They are also responsible for varying the converter parameters such that the space state definition in (3) represents the model $\hat{x} = A(p)x + B(p)u$, where $p = (V_i, Pot)$. Therefore, it is possible to state that the converter model depends on the input voltage and the output power. The number of vertices for the polytope is $n = 2^2 = 4$.

In order to verify the proposed model robustness, the closed-loop model with the uncertainties obtained during the process must be analysed. It is possible to state that a given system is robust if it is stable when uncertainties exist. For the proposed study, the chosen criterion is the complementary sensitivity. Shahian and Hassul (1993) say that a system is robust if it is able to reject disturbance and noise. The analysis tools used in this study are the complementary sensitivity transfer function T given by expression (20) and the multiplicative uncertainties.

$$T(z) = G(z)K(z)\left(I + G(z)K(z)\right)^{-1}, \tag{20}$$

where $G(z)$ is the model transfer function and $K(z)$ is the controller transfer function. Transfer function $T(z)$ is also known as the closed-loop transfer function of $\frac{y(z)}{r(z)}$, where $y(z)$ is the output and $r(z)$ is the set point tracking.

It is also worth to mention that the robustness analysis must ensure the stability margins against disturbances from additive and multiplicative uncertainties. According to Dorf and Bishop (1998) and Shahian and Hassul (1993), a discrete multiplicative uncertainty is given by

$$\Delta_m(z) = \left(\frac{\tilde{G}(z)}{G(z)} - 1\right), \tag{21}$$

where $\Delta_m(z)$ represents the symbol of multiplicative uncertainty and $\tilde{G}(z)$ is the transfer function considering the presence of uncertainty.

Dorf and Bishop (1998) and Shahian and Hassul (1993) say that disturbances are limited in magnitude, considering that G(z) and G(z) have the same number of poles in the left half plane (LHP). The stability is not supposed to change if

$$|\Delta_m(j\Omega)| < \left|\frac{1}{T(j\Omega)}\right|, \; \Omega \in [-\infty, \frac{\pi}{T_s}], \tag{22}$$

which is valid in the analysis with multiplicative uncertainties. The robustness analysis can be performed by using additive or multiplicative uncertainties. This paper considers the use of multiplicative uncertainties because it is easier to apply transfer function T in the feedback of state space systems. Besides, K(z) is not easy to obtain in servomechanisms with integral action. Therefore, this paper proposes the use of multiplicative uncertainties via parametric uncertainties.

2.3 LMI Concepts and LQR-LMI approach

A LMI has the following form(Boyd et al., 1994)

$$F(x) = F_0 + \sum_{k=1}^{m} x_i F_i, \geq 0, \tag{23}$$

where $x \in \mathbb{R}^m$ is the variable considering that the symmetric matrices $F_i = F_i^T \in \mathbb{R}^{n\times n}$ are given, $i = 1, \ldots, m$ are given and $F(x)$ is positive-definite. The LMI (23) is a convex constraint, i.e, $x|F(x) \geq 0$ is convex. Given $F(x) > 0$, a LMI Problem (LMIP) consists in determining x^{feas} such that $F(x^{feas}) > 0$ or determining if the LMI is infeasible. An example of LMIP (Linear Matrix Inequality Problem) is the "simultaneous discrete Lyapunov stability problem" given by

$$\begin{cases} \min_{P=P'} \; tr\{P\} \\ subject\,to: \quad \begin{array}{c} P > 0 \\ (A_i' P A_i - P) < -Q, \; i \in [1,n], \; n \in \mathbb{N} \end{array} \end{cases}, \tag{24}$$

where $P = P' > 0$ is the solution, $A_i \in \mathbb{R}^{n\times n}$ is given and $Q = Q' > 0$ is the desired equilibrium point.

Let the discrete LQR problem given by

$$J_{LQR} = \frac{1}{2}\sum_{i=1}^{\infty}[x^*(k)\,Qx(k) + u^*(k)\,Ru(k)], \tag{25}$$

where J_{LQR} is the performance index of a stable model, $x(k)$ and $u(k)$ are the state and signal control variables respectively,and $Q \geq 0$ and $R > 0$ are weighting matrices (Ogata, 1986).

From the continuous model obtained by Ghaoui et al. (1992), it is possible to obtain the same representation using algebraic transformations through the trace of matrices and Schur complement. Substituting $u(k) = -Kx(k)$ in (25) gives:

$$J_{LQR} = \frac{1}{2}\sum_{i=1}^{\infty}[x^*(k)\,(Q + K^* R K)\,x(k)]. \tag{26}$$

The trace of matrix is:

$$Tr\left\{J_{LQR}\right\} = Tr\left\{\frac{1}{2}\sum_{i=1}^{\infty}\left[x^*\left(k\right)\left(Q + K^*RK\right)x\left(k\right)\right]\right\}. \quad (27)$$

By using the trace properties, the following expression results:

$$Tr\left\{J_{LQR}\right\} = Tr\left\{\left(Q + K^*RK\right)\frac{1}{2}\sum_{i=1}^{\infty}\left[x^*\left(k\right)x\left(k\right)\right]\right\}. \quad (28)$$

It is possible to define $P = \frac{1}{2}\sum_{i=1}^{\infty}\left[x^*\left(k\right)x\left(k\right)\right]$ and $Y = KP$(Ghaoui et al., 1992) so that

$$Tr\left\{J_{LQR}\right\} = Tr\left\{\left(Q + K^*RK\right)P\right\} \quad (29)$$

$$Tr\left\{J_{LQR}\right\} = Tr\left\{\left(QP + Y^*P^{-1}RY\right)\right\} \quad (30)$$

$$Tr\left\{J_{LQR}\right\} = Tr\left\{\left(QP + Y^*\left(R^*\right)^{\frac{1}{2}}P^{-1}\left(R\right)^{\frac{1}{2}}Y\right)\right\} \quad (31)$$

thus

$$J_{LQR} = Tr\left\{QP\right\} + Tr\left\{Z\right\}, \quad (32)$$

where $Z > Y^*\left(R^*\right)^{\frac{1}{2}}P^{-1}\left(R\right)^{\frac{1}{2}}Y$ such that the minimization problem can be guaranteed. Applying the Schur complement to Z gives:

$$\begin{bmatrix} Z & \left(R^{1/2}Y\right)^* \\ \left(R^{1/2}Y\right) & P \end{bmatrix} > 0 \quad (33)$$

Therefore, the constraints for the discrete LQR using the LMI approach are given by

$$\begin{array}{l} \min\limits_{P,Y,Z}\ tr\left(QP\right) + tr\left(Z\right), \\ \begin{bmatrix} P & \left(A_iP - B_iY\right)' \\ \left(A_iP - B_iY\right) & P \end{bmatrix} > 0, \\ subject\ to \begin{bmatrix} Z & \left(R^{1/2}Y\right)^* \\ \left(R^{1/2}Y\right) & P \end{bmatrix} > 0,\ P > 0. \end{array}, \quad (34)$$

where $i = 1, \ldots, n$, and n is the quantity of system polytopes.

3 Simulation results

3.1 Simulation model

The simulation model was implemented and evaluated in MATLAB using the average space-state model defined by expressions (5), (6), (7) and (8). Besides, the discrete model was used so that:

$$A = \begin{bmatrix} -0.6310 & 1.5634 \\ -0.0497 & -0.6671 \end{bmatrix}, B = \begin{bmatrix} -65.0322 \\ 114.7816 \end{bmatrix}, \quad C = \begin{bmatrix} 0.0202 & 0.9962 \end{bmatrix}, D = -0.2575 \quad . \quad (35)$$

Besides, $T_s = 1ms$ and the worst case are also considered i.e. the system operation at full load. Therefore, the proposed represents a LTV system where $f(Pot(t), Vi(t), t)$, $Pot(t) \in [30, 60]W$ and $V_i(t) \in [24, 33.6]V$, such that $Pot(t)$ and $V_i(i)$ vary linearly from of lower to upper being the steps of T_s. Therefore, the plant in study is $A(Pot(t), V_i(t), t)$, $B(Pot(t), V_i(t), t)$, $C(Pot(t), V_i(t), t)$ and $D(Pot(t), V_i(t), t)$, becoming a LTV system according to Kothare et al. (1996).

The weighting matrices chosen were $Q = I_3$ and $R = 0.1$, such that both were used to work with discrete LQR typical and to work with LQR via LMI optimization. The gains of feedback control are given by

$$\begin{array}{l} K_{LMI} = \begin{bmatrix} -2.8501 \times 10^{-5} & -0.0039 \end{bmatrix}, \\ Ki_{LMI} = 0.0023 \end{array} \quad (36)$$

and

$$\begin{array}{l} K_{LQR} = \begin{bmatrix} 0.0013 & 0.0054 \end{bmatrix}, \\ Ki_{LQR} = 0.0031 \end{array} \quad (37)$$

where (36) and (37) represent the gain of the discrete LQR control via LMI and the gain of the discrete LQR control, respectively.

3.2 Obtained Results

Figure 3 shows the time response of the output voltage. It is possible to verify that the LQR-LMI control response is better than that obrained withe the traditional LQR control considering that both them are applied to a LTV system. The LQR-LMI control presents lower overshoot than that obtained with LQR control. Besides, the response is smoother in the first case.

Figure 4 shows the control signal response for both methods. It is possible to state that the LQR-LMI response is smoother than that of the LQR because the same behavior is verified in Figure 3. Therefore, under the same conditions in LTV mode using the same weighting matrices, the LQR-LMI control is able to maintain smoother but slower response than that of the LQR control.

Figure 5 shows that LQR-LMI control is more robust than LQR, despite both them are subjected to uncertainties. The uncertainty was obtained by using the nominal plant model given by (35) and its respective polytopes defined by the variation of parameters according to Table 2. Besides that, an additional margin of 50% is ensured in the model so that robustness is maintained. According to Figure 5, both control methods are robust, but LQRLMI performance is better. Figure 6 shows the root locus for LQR-LMI control. It is possible to notice that poles and zeros are within the unity circle, thus ensuring the closed-loop system stability.

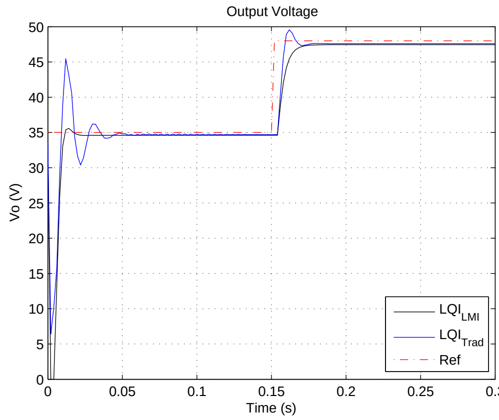

Figure 3: Output voltage.

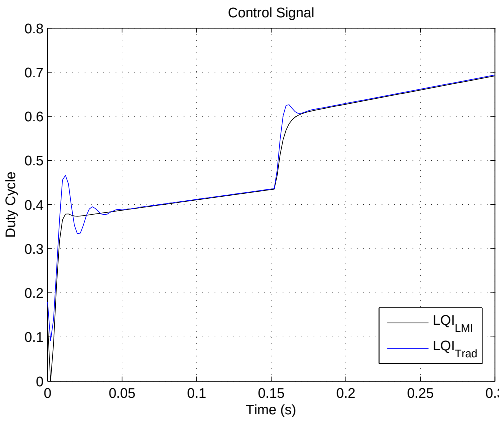

Figure 4: Control signal response.

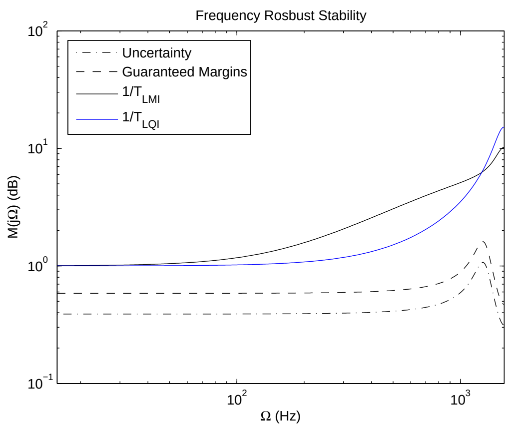

Figure 5: Robust Stability in the discrete frequency.

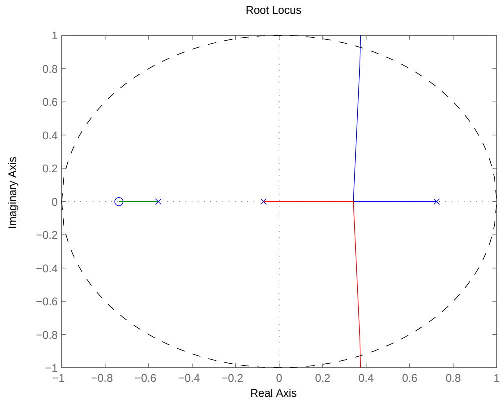

Figure 6: Discrete z-plane root locus.

4 Conclusions

This work has presented the analysis and study of a 3SSC-based boost converter using a discrete robust control approach. It has been shown that the discrete control methods present good performance, but LQR-LMI is more adequate from the point of view of the assessed response and robustness. The robust control with LMI has presented some improvement if compared with the optimal discrete LQR control.

Therefore, further effort must be made towards the study of LQR-LMI control. Current work in progress includes the experimental implementation of the aforementioned control techniques.

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