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ON THE GEOMETRY OF BACH-FLAT QUASI-EINSTEIN MANIFOLDS

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Prof. Dr. Halyson Irene Baltazar

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Resumo

Nesta dissertação, estudamos a rigidez geométrica em variedades quasi-Einstein compactas (M^n, g, f), $n \geqslant 4$ sob à condição do tensor de Bach nulo. Seguindo a abordagem metodológica de Chen e He [6], analisamos o comportamento do tensor de Weyl e da função potencial f nesses espaços. mostramos que a anulação do tensor de Bach restringe rigidamente a geometria da variedade, implicando que o espaço é localmente conformemente plano se $n = 4$, ou que possui tensor de Weyl harmônico e curvatura de Weyl radial nula se $n \geqslant 4$. Por fim, discutimos a classificação global decorrente dessas propriedades para o caso em que o parâmetro $m \neq 1$ e a constante λ é positiva.

Palavras-chave: Produto Warped, variedades quasi-Einstein, tensor de Bach, tensor de Weyl.

Abstract

In this dissertation, we study the geometric rigidity of compact quasi-Einstein manifolds (M^n, g, f), $n \geqslant 4$, under the Bach-flat condition. Following the approach of Chen and He [6], we analyze the behavior of the Weyl tensor and the potential function f on these spaces. It is shown that the vanishing of the Bach tensor rigidly restricts the geometry of the manifold, implying that the space is locally conformally flat if $n = 4$, or that it has a harmonic Weyl tensor and zero radial Weyl curvature if $n \geqslant 4$. Finally, we discuss the global classification resulting from these properties for the case where the parameter $m \neq 1$ and the constant λ is positive.

Keywords: Warped product, quasi-Einstein manifolds, Bach tensor, Weyl tensor.

Contents

Introduction

In the study of Riemannian geometry, particularly within the context of four-dimensional manifolds, it is a well-known fact that if a manifold is either Einstein or locally conformally flat, its Bach tensor vanishes identically. Motivated by this relationship, Chen and He [6] investigated the converse problem, seeking to determine the exact geometric conditions under which a Bach-flat manifold recovers these highly symmetric structures.

Following this line of inquiry, the central focus of this work is the study of a specific class of Riemannian manifolds subject to a vanishing Bach tensor, known as quasi-Einstein manifolds. A quasi-Einstein manifold, or m-quasi-Einstein manifold, is a complete Riemannian manifold (M^n, g) equipped with a smooth potential function $f \in C^\infty(M)$ that satisfies the $(\lambda, n+m)$-Einstein equation

$$Ric_f^m = R_{ij} + \nabla_i \nabla_j f - \frac{1}{m} \nabla_i f \nabla_j f = \lambda g_{ij}, \tag{1}$$

where λ and m are real constants. We call Ric_f^m in (1) the m-Bakry-Emery tensor. An important and motivating feature of this framework occurs when the parameter m is restricted to a positive integer. In this setting, $(\lambda, n+m)$-Einstein metrics on an n-dimensional manifold M^n correspond precisely to the base metrics of an $(n+m)$-dimensional Einstein warped product space. Specifically, the product manifold $(M^n \times F^m, \tilde{g})$ equipped with the warped metric

$$\tilde{g} = g + e^{-\frac{2f}{m}} g_F$$

is an Einstein manifold with Einstein constant λ where the space (F^m, g_F) is an Einstein manifold.

Therefore, throughout this work, we focus on building the foundational concepts and analytical tools necessary to establish classification and rigidity results for an important class of complete quasi-Einstein manifolds, those subject to a vanishing Bach tensor. On any Riemannian manifold (M^n, g) of dimension $n \geqslant 4$, the Bach tensor B is a symmetric, trace-free $(0,2)$-tensor whose components in local coordinates are defined by

$$B_{ij} = \frac{1}{n-3}\nabla_k \nabla_l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{ikjl}, \tag{2}$$

where ∇ denotes the covariant derivative, and W_{iklj} are the components of the Weyl curvature tensor.

Next, we focus our attention on presenting the main structural and classification results established throughout this work.

The first core result provides a definitive rigidity classification in the critical four-dimensional setting

Theorem 0.0.1. *Suppose (M^4, g, f) is a compact quasi-Einstein manifold with $m \neq 0, 1, -2$. If the Bach tensor B vanishes everywhere on M, then (M^4, g) is a locally conformally flat manifold.*

Theorem 0.0.1 arises as a direct consequence of a more general, higher-dimensional localization theorem combined with existing structural results detailed in [13].

Theorem 0.0.2. *Suppose (M^n, g, f) ($n \geqslant 4$) is a compact quasi-Einstein manifold with $m \neq 0, 1, 2-n$. If the Bach tensor B vanishes everywhere on M, then M has a harmonic Weyl tensor and satisfies*

$$W_{ijkl}\nabla_l f = 0.$$

Theorem 0.0.2 is conceptually analogous to the rigidity characterization for gradient Ricci solitons established by Cao and Chen [3].

By leveraging the global classification of warped product Einstein manifolds with a harmonic Weyl tensor and satisfying $W(\nabla f, \cdot, \cdot, \nabla f) = 0$ as proved in [13, Theorem 1.2], Theorem 0.0.2 yields the following definitive global classification parameter

Corollary 0.0.1. *Let $m \neq 1$ be a positive number. Suppose that (M^n, g, f) ($n \geqslant 4$) is a simply connected, compact quasi-Einstein manifold with $\lambda > 0$ and has a vanishing Bach tensor. Then (M^n, g, f) is either*

(i) *Einstein and the potential function f is constant, or*

(ii) *A warped product such that*

$$g = dt^2 + \psi^2(t) g_L \quad \textit{and} \quad f = f(t),$$

where g_L *is an Einstein metric with nonnegative Ricci curvature, and has constant curvature if* $n = 4$.

Remark 1. In the proof of [**13, Theorem 1.2**] , the authors assumed that $m > 1$.

Chapter 1

Background

1.1 Some Tensors concepts

In this section, we review some basic facts of Riemannian geometry and introduce the fundamental tensor fields and structural identities used throughout this work.

Definition 1.1.1. *Let V be a finite-dimensional vector space and V^* the dual space of V. A covariant s-tensor on V is a multilinear map*

$$T : V \times \cdots \times V_s \longrightarrow \mathbb{R}.$$

A contravariant r-tensor is a multilinear map

$$T : V^* \times \cdots \times V_r^* \longrightarrow \mathbb{R}.$$

A ***tensor of type*** (r, s) *is an s-covariant and r-contravariant tensor, that is, a multilinear map*

$$T : V^* \times \cdots \times V_r^* \times V \times \cdots \times V_s \longrightarrow \mathbb{R}.$$

In what follows, (M^n, g) will denote an n-dimensional Riemannian manifold with metric g and Levi-Civita connection ∇. The commutative ring of differentiable (or smooth, C^∞) functions on M will be denoted by $C^\infty(M)$. The space of smooth vector fields on M will be denoted by $\mathfrak{X}(M)$.

Definition 1.1.2. *A tensor field A on a manifold M^n is a tensor over the $C^\infty(M)$-module $\mathfrak{X}(M)$. Thus, an (r, s)-tensor A is a map*

$$A : \mathfrak{X}(M)^* \times \cdots \times \mathfrak{X}(M)^* \times \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$$

that is multilinear over C^∞. *More precisely,* A *is a* $C^\infty(M)$*-multilinear map that associates to each* $(r+s)$*-tuple* $(Y^1,\ldots,Y^r,X_1,\ldots,X_s)$ *a differentiable function*

$$f = A(Y^1,\ldots,Y^r,X_1,\ldots,X_s) : M^n \longrightarrow \mathbb{R}.$$

The position occupied by Y^i *is called the* i*-th contravariant entry, and the position occupied by* X_j *is called the* j*-th covariant entry. In this manner,* $(0,s)$*-tensors are said to be covariant and* $(r,0)$*-tensors are said to be contravariant. We denote by* $\mathfrak{I}^r_s(M)$ *the module of all* (r,s)*-tensors over* $C^\infty(M)$.

The following definition establishes how the product of two tensors is constructed.

Definition 1.1.3. *Let* $A \in \mathfrak{I}^r_s(M)$ *and* $B \in \mathfrak{I}^{r'}_{s'}(M)$. *The tensor product of the tensor* A *with the tensor* B *is the tensor*

$$A \otimes B : \mathfrak{X}^*(M)^{r+r'} \times \mathfrak{X}(M)^{s+s'} \longrightarrow C^\infty(M)$$

defined by

$$\begin{aligned}&(A\otimes B)(Y_1,\ldots,Y_{r+r'},X_1,\ldots,X_{s+s'})\\ &= A(Y_1,\ldots,Y_r,X_1,\ldots,X_s)\,B(Y_{r+1},\ldots,Y_{r+r'},X_{s+1},\ldots,X_{s+s'}).\end{aligned}$$

If $r' = s' = 0$, *then* B *is a function* $f \in C^\infty(M)$. *In this case, we define*

$$A \otimes f = f \otimes A = fA.$$

Moreover, if A *is a* $(0,0)$*-tensor, then the tensor product reduces to multiplication in* $C^\infty(M)$.

Thus, throughout this work, whenever we refer to an (r,s)*-tensor, we shall regard it as a multilinear map. Furthermore, throughout the text, we shall adopt the Einstein summation convention, namely, the summation symbol is omitted whenever an upper index and a lower index are repeated. For instance,*

$$y_i = \sum_{i=1}^{n} x_i^j E_j.$$

is equivalent to

$$y_i = x_i^j E_j$$

1.2 Differential Operators

Definition 1.2.1. *We define the* ***covariant derivative*** *of a* $(1, s)$*-tensor* T *to be the* $(1, s+1)$*-tensor*

$$\nabla T : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s+1 \text{ times}} \longrightarrow \mathfrak{X}(M)$$

given by

$$\begin{aligned} \nabla T(X, Y_1, \dots, Y_s) &= (\nabla_X T)(Y_1, \dots, Y_s) \\ &= \nabla_X (T(Y_1, \dots, Y_s)) - \sum_{i=1}^{s} T(Y_1, \dots, \nabla_X Y_i, \dots, Y_s). \end{aligned}$$

Definition 1.2.2. *Let* $f : M \longrightarrow \mathbb{R}$ *be a differentiable function. The* ***gradient*** *of* f *is the differentiable vector field* ∇f *defined on* M *by*

$$\langle \nabla f, X \rangle = X(f),$$

for all $X \in \mathfrak{X}(M)$.

An alternative characterization of this definition is provided by the following proposition.

Proposition 1.2.1. *Let* $f : M \longrightarrow \mathbb{R}$ *be a smooth function and* $\{e_1, e_2, \dots, e_n\}$ *an orthonormal frame on an open neighborhood* $U \subset M$*. Then, on* U *we have*

$$\nabla f = \sum_{i=1}^{n} e_i(f) e_i.$$

Proof. On U we can write $\nabla f = \sum_{i=1}^{n} a_i e_i$. Thus, it can be concluded by observing that

$$e_j(f) = a_j.$$

□

Furthermore, some fundamental properties regarding the gradient of the potential function are detailed in the sequel.

Proposition 1.2.2. *If* $f, g : M^n \longrightarrow \mathbb{R}$ *are smooth functions, then*

(i) $\nabla(f + g) = \nabla f + \nabla g$;

(ii) $\nabla(fg) = g\nabla f + f\nabla g$.

Proof. It suffices to observe that, for any smooth vector field X on M, it follows directly from the definition of the gradient that

$$\begin{aligned}\langle \nabla(f+g), X\rangle &= X(f+g) = X(f) + X(g)\\ &= \langle \nabla f, X\rangle + \langle \nabla g, X\rangle\\ &= \langle \nabla f + \nabla g, X\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle \nabla(fg), X\rangle &= X(fg) = gX(f) + fX(g)\\ &= \langle g\nabla f, X\rangle + \langle f\nabla g, X\rangle\\ &= \langle g\nabla f + f\nabla g, X\rangle.\end{aligned}$$

□

We now define the divergence

Definition 1.2.3. *Let X be a smooth vector field on M^n. The **divergence** of X is the smooth function **div**$X : M^n \longrightarrow \mathbb{R}$, given for $p \in M$ by*

$$(\mathit{div}X)(p) = \mathit{tr}\{v \mapsto (\nabla_v X)(p)\}, \tag{1.1}$$

*where $v \in T_pM$ and **tr** denotes the trace of the linear operator enclosed in braces.*

Similarly to the previous definition, the divergence of a $(1,r)$-tensor T is defined as the $(0,r)$-tensor

$$\begin{aligned}(\mathit{div}T)(v_1,\ldots,v_r) &= \mathit{tr}\{w \mapsto (\nabla_w T)(v_1,\ldots,v_r)\}\\ &= g^{jk}\nabla_j T_k(v_1,\ldots,v_r).\end{aligned}$$

In particular, if T is a $(0,2)$-tensor, then in local coordinates we have

$$(\mathit{div}T)_i = g^{jk}\nabla_k T_{ij}.$$

To illustrate the role of the divergence as a core technical tool, we outline some of its fundamental properties in the sequel.

Proposition 1.2.3. *If X, Y are smooth vector fields on M^n and $f : M^n \longrightarrow \mathbb{R}$ is a smooth function, then*

(i) $div(X+Y) = divX + divY$;

(ii) $div(fX) = fdivX + \langle \nabla f, X\rangle$.

Proof. Consider X and Y to be smooth vector fields defined on M^n, and let $\{e_1, \dots, e_n\}$ be an orthonormal frame on an open neighborhood $U \subset M$. Thus, we have

$$\begin{aligned}\mathrm{div}(X+Y) &= \sum_{i=1}^{n} \langle \nabla_{e_i}(X+Y), e_i\rangle \\ &= \sum_{i=1}^{n} \langle \nabla_{e_i}X + \nabla_{e_i}Y, e_i\rangle \\ &= \sum_{i=1}^{n} \langle \nabla_{e_i}X, e_i\rangle + \sum_{i=1}^{n} \langle \nabla_{e_i}Y, e_i\rangle \\ &= \mathrm{div}X + \mathrm{div}Y,\end{aligned}$$

and

$$\begin{aligned}\mathrm{div}(fX) &= \sum_{i=1}^{n} \langle \nabla_{e_i}(fX), e_i\rangle \\ &= \sum_{i=1}^{n} \langle e_i(f)X + f\nabla_{e_i}X, e_i\rangle \\ &= \sum_{i=1}^{n} \langle e_i(f)e_i, X\rangle + f\sum_{i=1}^{n} \langle \nabla_{e_i}X, e_i\rangle \\ &= \langle \nabla f, X\rangle + f\mathrm{div}X.\end{aligned}$$

□

As a consequence of this framework, we recall the classical Divergence Theorem.

Theorem 1.2.1 (Divergence Theorem). *Let M^n be a compact oriented Riemannian manifold and $X \in \mathfrak{X}(M)$. If the boundary of M is equipped with the orientation and the metric induced by the inclusion $j : \partial M \longrightarrow M$, and υ denotes the outward unit normal vector to M along ∂M, then*

$$\int_M (divX)\, dM = \int_{\partial M} \langle X, \upsilon\rangle\, d(\partial M).$$

The aforementioned theorem provides a key technical tool for establishing our main results.

Definition 1.2.4. *Let* f *be a differentiable function on* M. *The* ***Laplacian*** *of* f *is the function* $\Delta f : M \longrightarrow \mathbb{R}$ *given by*

$$\Delta f = div(\nabla f).$$

In local coordinates,

$$\Delta f = div(\nabla f) = g^{ij}\nabla_i \nabla_j f.$$

Definition 1.2.5. *Let* $f : M \longrightarrow \mathbb{R}$ *be a differentiable function. We define the* ***Hessian*** *of* f *as the* $(1,1)$*-tensor given by*

$$(\mathrm{Hess} f)(X) = \nabla_X \nabla f, \ \ X \in \mathfrak{X}(M),$$

or as the $(0,2)$*-tensor given by*

$$\mathrm{Hess}\, f(X,Y) = \langle \nabla_X \nabla f, Y\rangle = \nabla^2_{X,Y}(f), \ \ X,Y \in \mathfrak{X}(M).$$

In local coordinates,

$$\mathrm{Hess}\, f(\partial_i, \partial_j) = \nabla_i \nabla_j f.$$

This proposition provides a crucial analytical tool for studying our main tensors and fundamental equations. In particular, it shows that the Laplacian of f is simply the trace of its Hessian.

Proposition 1.2.4. *Let* $f : M \longrightarrow \mathbb{R}$ *be a differentiable function. In local coordinates, the Laplacian of* f *is given by the trace of its Hessian matrix*

$$\Delta f = g^{ij}\nabla_i \nabla_j f.$$

Proof. It suffices to verify the equality locally at each point $p \in M$. Let (U, x^i) be a local coordinate chart around p. By the definition of the Laplacian in coordinates and the local expression for the Hessian of a function, we have

$$\begin{aligned}\Delta f(p) &= \mathrm{div}(\nabla f)(p)\\ &= g^{ij}\nabla_i (\nabla f)_j (p)\\ &= g^{ij}\nabla_i \nabla_j f(p).\end{aligned}$$

Since the trace of the $(0,2)$-tensor $\nabla^2 f$ with respect to the metric g is precisely the contraction $g^{ij}\nabla_i\nabla_j f$, the result follows. $\square$

Recall that an orthonormal frame $\{E_1, \dots, E_n\}$ on an open set $U \subset M$ is said to be **geodesic** at a point $p \in U$ if $(\nabla_{E_i} E_j)(p) = 0$ for all $1 \leqslant i, j \leqslant n$.

Here, we define the Riemann curvature tensor, a central object in this framework that serves as a bridge between the algebra of tensors and the behavior of differential operators.

Definition 1.2.6. *Let (M^n, g) be a Riemannian manifold. The **Riemann curvature tensor** is the $(1,3)$-tensor*

$$R_m : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M),$$

given by

$$R_m(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z,$$

for all $X, Y, Z \in \mathfrak{X}(M)$. In local coordinates,

$$R_m(\partial_i, \partial_j)\partial_k = R_{lijk}\, g^{lm} \partial_m.$$

Furthermore, whenever necessary, we shall adopt the notation $\partial_i = \frac{\partial}{\partial x^i}$.

Utilizing the metric tensor, we can interpret R_m as a $(0,4)$-tensor, defined by

$$R_m : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M),$$

where

$$R_m(X, Y, Z, W) = g(R_m(X, Y)Z, W).$$

In local coordinates, this yields

$$R_m(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl}.$$

To better understand the structural behavior of this tensor, we record its key properties below.

Proposition 1.2.5. *The Riemann curvature tensor satisfies the following algebraic and differential properties*

(i) $Rm(X, Y, W, Z) = -Rm(Y, X, W, Z) = Rm(Y, X, Z, W)$.

(ii) $Rm(X, Y, W, Z) = Rm(W, Z, X, Y)$.

(iii) ***First Bianchi Identity***

$$Rm(X, Y, W, Z) + Rm(Y, W, X, Z) + Rm(W, X, Y, Z) = 0.$$

In local coordinates,

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0;$$

(iv) ***Second Bianchi Identity***

$$(\nabla_W Rm)(X, Y, Z, U) + (\nabla_Y Rm)(W, X, Z, U) + (\nabla_X Rm)(Y, W, Z, U) = 0.$$

In local coordinates,

$$\nabla_l R_{ijks} + \nabla_j R_{liks} + \nabla_i R_{jlks} = 0.$$

Proof. For the proof of this theorem, we refer the reader to [11]. □

Before working with the next definitions and their properties, we must first review an essential tool

Proposition 1.2.6. *Let* T *be a* $(0, 2)$*-tensor field and* g *be a Riemannian metric. Then, the inner product of* g *and* T *satisfies*

$$\langle g, T \rangle = tr\{T\}.$$

Proof. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame on the manifold. Expressing the components of the $(0, 2)$-tensor T with respect to this basis as $T_{ij} = T(e_i, e_j)$, we have

$$\langle g, T \rangle = \sum_{i,j=1}^{n} g_{ij} T_{ij}.$$

Since $\{e_i\}$ is an orthonormal frame, the components of the metric tensor reduce to the Kronecker delta, $g_{ij} = \delta_{ij}$. Substituting this into the summation yields

$$\langle g, T \rangle = \sum_{i,j=1}^{n} \delta_{ij} T_{ij} = \sum_{i=1}^{n} T_{ii} = \mathrm{tr}_g\{T\}.$$

Therefore, we conclude that $\langle g, T \rangle = \mathrm{tr}\{T\}$. □

Definition 1.2.7. *Given two non-zero and linearly independent vectors* X *and* Y *in* T_pM*, we define the* ***sectional curvature*** *of the plane* σ *spanned by* X *and* Y *by*

$$K(\sigma) = K(X, Y) = \frac{Rm(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$

Note that the following definitions immediately come from the contraction of the Riemann curvature tensor.

Definition 1.2.8. *We define the **Ricci curvature tensor** as the* $(0,2)$*-tensor*

$$Ric : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M),$$

given by the trace of the Riemann curvature tensor, i.e.,

$$Ric(X, Z) = tr\{Y \mapsto Rm(X, Y)Z\},$$

where $X, Y, Z \in \mathfrak{X}(M)$*. In local coordinates,*

$$Ric(\partial_i, \partial_k) = R_{ik} = g^{jl} R_{ijkl}.$$

Definition 1.2.9. *The **scalar curvature** of a manifold is the function* $R : M \longrightarrow \mathbb{R}$ *given by the trace of the Ricci tensor:*

$$R = tr\, Ric.$$

In local coordinates,

$$R = g^{ik} R_{ik}.$$

The results above are recorded for future reference, as they serve as core tools in our subsequent proofs.

Proposition 1.2.7. *Let* (M^n, g) *be a Riemannian manifold. The following geometric identities hold*

(i) ***Contracted Ricci Identity***

$$\nabla_j R_{ik} - \nabla_i R_{jk} = g^{ls} \nabla_l R_{ijks};$$

(ii) ***Twice-Contracted Second Bianchi Identity***

$$g^{jk} \nabla_j R_{ik} = \frac{1}{2} \nabla_i R.$$

Proof. To prove the first item, we take the trace of the second Bianchi identity by contracting it with the metric components g^{ls}, which yields

$$\begin{aligned} 0 &= g^{ls} \nabla_l R_{jiks} + g^{ls} \nabla_j R_{ilks} + g^{ls} \nabla_i R_{ljks} \\ &= g^{ls} \nabla_l R_{jiks} + \nabla_j \left(g^{ls} R_{ilks} \right) - \nabla_i \left(g^{ls} R_{jlks} \right) \\ &= -g^{ls} \nabla_l R_{ijks} + \nabla_j R_{ik} - \nabla_i R_{jk}. \end{aligned}$$

Therefore, isolating the divergence term, we obtain

$$g^{ls}\nabla_l R_{ijks} = \nabla_j R_{ik} - \nabla_i R_{jk}.$$

Furthermore, the aforementioned equation can be rewritten by rearranging the index tracks to isolate the derivative of the Ricci tensor

$$\nabla_i R_{jk} = \nabla_j R_{ik} + g^{ls}\nabla_l R_{jiks}.$$

We now proceed to show the second item. Taking the trace of the equation obtained in the first item by contracting it with g^{jk}, we find that

$$\nabla_i \left(g^{jk} R_{jk}\right) = g^{jk}\nabla_j R_{ik} + g^{ls}\nabla_l \left(g^{jk} R_{jiks}\right).$$

By substituting the definition of the scalar curvature $R = g^{jk}R_{jk}$ and contracting the Riemann tensor in the final term, this reduces to

$$\nabla_i R = g^{jk}\nabla_j R_{ik} + g^{ls}\nabla_l R_{is}.$$

Consequently, by executing a standard relabeling of the dummy indices (replacing s with k and l with j in the last term), we arrive at

$$\nabla_i R = 2g^{jk}\nabla_j R_{ik},$$

which completes the proof. □

Proposition 1.2.8. (Ricci Identity) *Let* M^n *be a Riemannian manifold and* $f : M^n \longrightarrow \mathbb{R}$ *a smooth function. For any vector fields* $X, Y, Z \in \mathfrak{X}(M)$*, the following identity holds*

$$(\nabla_X \mathit{Hess}\, f)(Y, Z) - (\nabla_Y \mathit{Hess}\, f)(X, Z) = \langle \mathit{Rm}(X, Y)Z, \nabla f\rangle.$$

In local coordinates,

$$\nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = R_{ijkl}\nabla_l f.$$

Proof. By definition, it follows

$$\begin{aligned}(\nabla_X \mathrm{Hess}\, f)(Y, Z) &= X(\mathrm{Hess}\, f(Y, Z)) - \mathrm{Hess}\, f(\nabla_X Y, Z) - \mathrm{Hess}\, f(Y, \nabla_X Z)\\ &= X\langle \nabla_Y \nabla f, Z\rangle - \langle \nabla_{\nabla_X Y}\nabla f, Z\rangle - \langle \nabla_Y \nabla f, \nabla_X Z\rangle\\ &= \langle \nabla_X \nabla_Y \nabla f, Z\rangle - \langle \nabla_{\nabla_X Y}\nabla f, Z\rangle,\end{aligned}$$

that is,

$$(\nabla_X \mathrm{Hess}\, f)(Y, Z) = \langle \nabla_{\nabla_X Y} Z, \nabla f \rangle - \langle \nabla_X \nabla_Y Z, \nabla f \rangle. \tag{1.2}$$

Analogously,

$$(\nabla_Y \mathrm{Hess}\, f)(X, Z) = \langle \nabla_{\nabla_Y X} Z, \nabla f \rangle - \langle \nabla_Y \nabla_X Z, \nabla f \rangle. \tag{1.3}$$

Thus, by subtracting (1.3) from (1.2), we obtain

$$(\nabla_X \mathrm{Hess}\, f)(Y, Z) - (\nabla_Y \mathrm{Hess}\, f)(X, Z) = \langle \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z, \nabla f \rangle. \tag{1.4}$$

Now, from the definition of the Riemann curvature tensor, we conclude that

$$(\nabla_X \mathrm{Hess}\, f)(Y, Z) - (\nabla_Y \mathrm{Hess}\, f)(X, Z) = \langle \mathrm{Rm}(X, Y)Z, \nabla f \rangle.$$

Thus, we have proven the desired result. □

At this stage, we establish the foundational tensor definitions and their respective expressions in local coordinates that will be utilized throughout this dissertation. For a Riemannian manifold of dimension $n \geqslant 4$, the local components of the **Weyl curvature tensor**, denoted by W_{ijkl}, are defined via the algebraic decomposition of the Riemann curvature tensor as follows

$$R_{ijkl} = W_{ijkl} + \frac{1}{(n-2)} (\mathrm{Ric} \odot g)_{ijkl} - \frac{R}{2(n-1)(n-2)} (g \odot g)_{ijkl}, \tag{1.5}$$

where R_{ijkl} are the components of the Riemann tensor, Ric are the components of the Ricci tensor, and g_{ij} represents the Riemannian metric.

Explicitly, for any two symmetric $(0, 2)$-tensors with local components S_{ij} and T_{ij}, their Kulkarni–Nomizu product components $(S \odot T)_{ijkl}$ are given by

$$(S \odot T)_{ijkl} = (S_{ik} T_{jl} + S_{jl} T_{ik} - S_{il} T_{jk} - S_{jk} T_{il}).$$

Furthermore, the orthogonal decomposition of the Riemann curvature tensor could be expressed as

$$R_{ijkl} = W_{ijkl} + (A \odot g)_{ijkl}, \tag{1.6}$$

where A_{ij} denotes the components of the **Schouten tensor**, defined by

$$A_{ij} = \frac{1}{n-2} \left(R_{ij} - \frac{R}{2(n-1)} g_{ij} \right). \tag{1.7}$$

Consequently, equation (1.6) can be rewritten as

$$\begin{aligned} R_{ijkl} &= W_{ijkl} + \frac{1}{n-2}(\mathrm{Ric} \odot g)_{ijkl} - \frac{R}{2(n-1)(n-2)}(g \odot g)_{ijkl} \\ &= W_{ijkl} + \frac{1}{(n-2)}\left(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\right) \\ &\quad - \frac{R}{(n-1)(n-2)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right), \end{aligned} \tag{1.8}$$

Connecting the idea

Proposition 1.2.9. *The Weyl tensor satisfies the following properties*

(i) W *is trace-free;*

(ii) $W_{ijkl} = -W_{jikl} = W_{ijlk}$*;*

(iii) *The tensor* W *satisfies the first Bianchi identity, that is,*

$$W_{ijkl} + W_{jkil} + W_{kijl} = 0. \tag{1.9}$$

Proof. To prove item (i), we compute the trace of the Weyl tensor with respect to the indices i and k by contracting with the inverse metric g^{ik}

$$\begin{aligned} g^{ik}W_{ijkl} = g^{ik}R_{ijkl} &- \frac{1}{n-2}g^{ik}\left(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\right) \\ &+ \frac{R}{(n-1)(n-2)}g^{ik}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right). \end{aligned}$$

Recalling the standard contractions $g^{ik}R_{ijkl} = R_{jl}$, $g^{ik}R_{ik} = R$, $g^{ik}g_{ik} = n$, $g^{ik}R_{il} = R^k_l$, and $g^{ik}g_{il} = \delta^k_l$, we can substitute these relations directly

$$\begin{aligned} g^{ik}W_{ijkl} &= R_{jl} - \frac{1}{n-2}\left(Rg_{jl} + nR_{jl} - R_{jl} - R_{jl}\right) \\ &\quad + \frac{R}{2(n-1)(n-2)}\left(ng_{jl} - g_{jl}\right) \\ &= R_{jl} - \frac{1}{n-2}\left(Rg_{jl} + (n-2)R_{jl}\right) + \frac{R}{(n-1)(n-2)}(n-1)g_{jl} \\ &= R_{jl} - \frac{R}{n-2}g_{jl} - R_{jl} + \frac{R}{(n-2)}g_{jl}. \end{aligned}$$

Combining the remaining metric scalar terms yields

$$g^{ik}W_{ijkl} = (1-1)\,R_{jl} + \left(-\frac{1}{n-2} + \frac{1}{n-2}\right)Rg_{jl} = 0.$$

Analogously, contracting over any other pair of indices demonstrates that the Weyl tensor is completely trace-free.

For item (ii), using the relation $R_{ijkl} = W_{ijkl} + (A \odot g)_{ijkl}$, we observe that

$$\begin{aligned} W_{ijkl} &= R_{ijkl} - (A \odot g)_{ijkl} \\ &= -R_{jikl} - (A_{ik}g_{jl} + A_{jl}g_{ik} - A_{il}g_{jk} - A_{jk}g_{il}) \\ &= -R_{jikl} + A_{jk}g_{il} + A_{il}g_{jk} - A_{jl}g_{ik} - A_{ik}g_{jl} \\ &= -\big(R_{jikl} - (A \odot g)_{jikl}\big) \\ &= -W_{jikl}. \end{aligned}$$

Similarly, for the second symmetry relation

$$\begin{aligned} -W_{jikl} &= -\big(R_{jikl} - (A \odot g)_{jikl}\big) \\ &= -R_{jikl} + A_{jk}g_{il} + A_{il}g_{jk} - A_{jl}g_{ik} - A_{ik}g_{jl} \\ &= R_{jilk} - (A_{jl}g_{ik} + A_{ik}g_{jl} - A_{jk}g_{il} - A_{il}g_{jk}) \\ &= R_{jilk} - (A \odot g)_{jilk} \\ &= W_{jilk}, \end{aligned}$$

which completes the proof of item (ii).

Substituting the orthogonal decomposition equation into the algebraic first Bianchi identity for the Riemann tensor yields

$$\begin{aligned} 0 &= R_{ijkl} + R_{jkil} + R_{kijl} \\ &= \big(W_{ijkl} + (A \odot g)_{ijkl}\big) + \big(W_{jkil} + (A \odot g)_{jkil}\big) + \big(W_{kijl} + (A \odot g)_{kijl}\big). \end{aligned}$$

Rearranging the terms to isolate the Weyl tensor components, we obtain

$$\begin{aligned} W_{ijkl} + W_{jkil} + W_{kijl} &= -(A \odot g)_{ijkl} - (A \odot g)_{jkil} - (A \odot g)_{kijl} \\ &= -\,(A_{ik}g_{jl} + A_{jl}g_{ik} - A_{il}g_{jk} - A_{jk}g_{il}) \\ &\quad - (A_{ji}g_{kl} + A_{kl}g_{ji} - A_{jl}g_{ki} - A_{ki}g_{jl}) \\ &\quad - (A_{kj}g_{il} + A_{il}g_{kj} - A_{kl}g_{ij} - A_{ij}g_{kl}) \\ &= 0, \end{aligned}$$

where all terms cancel out identically due to the pairwise symmetry of the Riemannian metric ($g_{ij} = g_{ji}$) and the Schouten tensor ($A_{ij} = A_{ji}$). This concludes the proof of item (iii). □

We say that a Riemannian metric is *Bach-flat* when its Bach tensor vanishes identically, that is, $B_{ij} = 0$.

Definition 1.2.10. *On a Riemannian manifold* (M^n, g) *with dimension* $n \geqslant 3$*, the* ***Cotton tensor*** *is the* $(0,3)$*-tensor whose local components* C_{ijk} *are defined by*

$$\begin{aligned} C_{ijk} &= \nabla_i A_{jk} - \nabla_j A_{ik} \\ &= \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} \left((\nabla_i R) g_{jk} - (\nabla_j R) g_{ik} \right). \end{aligned} \tag{1.10}$$

Having defined the Cotton tensor, we can now establish an important relationship between it and the Weyl tensor as follows

Proposition 1.2.10. *For a Riemannian manifold of dimension* $n \geqslant 4$*, the Cotton tensor components* C_{ijk} *can be expressed, up to a constant factor, as the divergence of the Weyl tensor*

$$C_{ijk} = -\frac{n-2}{n-3} \nabla_l W_{ijkl}. \tag{1.11}$$

Proof. To prove this proposition, consider the Riemann decomposition

$$\begin{aligned} R_{ijkl} = W_{ijkl} &+ \frac{1}{n-2} \left(R_{ik} g_{jl} + R_{jl} g_{ik} - R_{il} g_{jk} - R_{jk} g_{il} \right) \\ &- \frac{R}{(n-1)(n-2)} \left(g_{ik} g_{jl} - g_{il} g_{jk} \right). \end{aligned}$$

Next, taking the trace by contractig with ∇_l, and using the results from **Proposition 1.2.7**, it follows that

$$\begin{aligned} \nabla_j R_{ik} - \nabla_i R_{jk} = \nabla_l W_{ijkl} &+ \frac{1}{n-2} \left(\nabla_j R_{ik} - \nabla_i R_{jk} \right) + \frac{1}{2(n-2)} \left(\nabla_j R g_{ik} - \nabla_i R g_{jk} \right) \\ &- \frac{1}{(n-1)(n-2)} \left(\nabla_j R g_{ik} - \nabla_i R g_{jk} \right). \end{aligned}$$

Putting similar terms together yields

$$\begin{aligned} \nabla_l W_{ijkl} &= \left(1 - \frac{1}{n-2} \right) \left(\nabla_j R_{ik} - \nabla_i R_{jk} \right) \\ &\quad - \left(\frac{1}{2(n-2)} - \frac{1}{(n-1)(n-2)} \right) \left(\nabla_j R g_{ik} - \nabla_i R g_{jk} \right) \\ &= -\frac{n-3}{n-2} \left(\left(\nabla_i R_{jk} - \nabla_j R_{ik} \right) - \frac{1}{2(n-1)} \left(\nabla_i R g_{jk} - \nabla_j R g_{ik} \right) \right) \\ &= -\frac{n-3}{n-2} C_{ijk}. \end{aligned}$$

and finally, we get

$$C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl},$$

as desired. □

Furthermore, the Cotton tensor satisfies these identities.

Proposition 1.2.11 (Properties of the Cotton tensor)**.** *The Cotton tensor satisfies the following geometric properties*

(i) *The Cotton tensor is skew-symmetric in its first two indices, that is,*

$$C_{ijk} = -\,C_{jik}; \tag{1.12}$$

(ii) *The Cotton tensor is totally trace-free, meaning that*

$$g^{ij}C_{ijk} = g^{ik}C_{ijk} = g^{jk}C_{ijk} = 0. \tag{1.13}$$

Proof. To prove item (i), we apply the definition of the Cotton tensor

$$\begin{aligned} C_{ijk} &= (n-2)\left(\nabla_i A_{jk} - \nabla_j A_{ik}\right) \\ &= -(n-2)\left(\nabla_j A_{ik} - \nabla_i A_{jk}\right) \\ &= -C_{jik}, \end{aligned}$$

which establishes the requested skew-symmetry.

To prove item (ii), we utilize the relation established in equation (1.11)

$$g^{ij}C_{ijk} = -\frac{n-2}{n-3}\nabla^l\left(g^{ij}W_{ijkl}\right) = 0,$$

which vanishes identically because the Weyl tensor is trace-free on its first two indices ($g^{ij}W_{ijkl} = 0$).

Similarly, contracting the metric across the remaining index tracks yields

$$g^{ik}C_{ijk} = -\frac{n-2}{n-3}\nabla^l\left(g^{ik}W_{ijkl}\right) = 0,$$

which vanishes because the Weyl tensor is trace-free on its first and third indices ($g^{ik}W_{ijkl} = 0$). □

Definition 1.2.11. *A Riemannian manifold* (M^n, g) *has a harmonic Weyl tensor if the Cotton tensor vanishes.*

1.3 Einstein Manifold

In this section, we discuss some concepts and results about Einstein manifolds.

Definition 1.3.1. *A Riemannian manifold* (M^n, g) *is called an Einstein manifold if its Ricci tensor is a multiple of the metric* g*, that is,*

$$\mathrm{Ric}(X, Y) = \lambda g(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$*, where* $\lambda \colon M \to \mathbb{R}$ *is a smooth function. In local coordinates, this condition is expressed as*

$$R_{ij} = \lambda g_{ij}. \tag{1.14}$$

Remark 2. Note that by taking the trace of equation (1.14), it follows that $R = \lambda n$, which implies $\lambda = \frac{R}{n}$. Consequently, M is an Einstein manifold if and only if

$$\mathrm{Ric} = \frac{R}{n} g.$$

Observe that if a (M^n, g) be a connected Einstein manifold of dimension $n \geqslant 3$, then λ is constant. Indeed, since the manifold is Einstein, we have $R_{ik} = \lambda g_{ik}$ and, consequently, $R = \lambda n$. Thus, applying the twice-contracted second Bianchi identity, we obtain

$$\begin{aligned} n\nabla_i \lambda = \nabla_i R &= 2g^{jk}\nabla_j R_{ik} \\ &= 2\nabla^k R_{ik} \\ &= 2\nabla^k(\lambda g_{ik}) \\ &= 2\nabla_i \lambda, \end{aligned}$$

which implies

$$(n-2)\nabla_i \lambda = 0.$$

Since $n \geqslant 3$, it follows that $\nabla_i \lambda = 0$. Given that the manifold M is connected, we conclude that λ is constant.

Definition 1.3.2. *On a Riemannian manifold* (M^n, g), $(n \geqslant 4)$*, the* ***Bach tensor*** *is the symmetric* $(0, 2)$*-tensor whose local components* B_{ij} *are defined by*

$$B_{ij} = \frac{1}{n-3}\nabla_k \nabla_l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{ikjl}, \tag{1.15}$$

where $\nabla_k \nabla_l$ *denotes the second covariant derivative operator with raised indices.*

The following remark establishes a key identity that will be utilized in the proof of Theorem 4.1.1

Remark 3. Using the previous result we could rewrite the Bach tensor as being

$$\begin{aligned} B_{ij} &= \frac{1}{n-3}\nabla_k\nabla_l W_{ikjl} + \frac{1}{n-2}R_{kl}W_{ikjl} \\ &= -\frac{1}{n-2}\nabla_k C_{ikj} + \frac{1}{n-2}R_{kl}W_{ikjl} \\ &= \frac{1}{n-2}\left(\nabla_k C_{kij} + R_{kl}W_{ikjl}\right). \end{aligned}$$

1.4 Warped Product

We now present the core concepts and properties of warped products that constitute the mathematical foundation for our main theorems.

Let M_1 and M_2 be Riemannian manifolds. The product manifold structure on $M_1 \times M_2$ ensures that the canonical projections $\pi_{M_1} : M_1 \times M_2 \to M_1$ and $\pi_{M_2} : M_1 \times M_2 \to M_2$ are smooth functions. The usual product metric on $M_1 \times M_2$ is given by

$$\langle \cdot, \cdot \rangle = \pi^*_{M_1}(\langle \cdot, \cdot \rangle_{M_1}) + \pi^*_{M_2}(\langle \cdot, \cdot \rangle_{M_2}).$$

We now define a new Riemannian metric, which will be utilized throughout this work. Let $(M_1, \langle \cdot, \cdot \rangle_{M_1})$ and $(M_2, \langle \cdot, \cdot \rangle_{M_2})$ be Riemannian manifolds, and let $f : M_1 \to \mathbb{R}$ be a positive smooth function on M_1. On the product manifold $M_1 \times M_2$, the **warped product metric** is defined by

$$\langle \cdot, \cdot \rangle = \pi^*_{M_1}(\langle \cdot, \cdot \rangle_{M_1}) + (f \circ \pi_{M_1})^2 \pi^*_{M_2}(\langle \cdot, \cdot \rangle_{M_2}). \tag{1.16}$$

Explicitly, since the tangent space splits as $T_{(p,q)}(M_1 \times M_2) \cong T_pM_1 \oplus T_qM_2$, given vectors $u = (u_{M_1}, u_{M_2})$ and $v = (v_{M_1}, v_{M_2})$ in $T_{(p,q)}(M_1 \times M_2)$, equation (1.16) can be evaluated pointwise as

$$\langle u, v \rangle = \langle u_{M_1}, v_{M_1} \rangle_{M_1} + f(p)^2 \langle u_{M_2}, v_{M_2} \rangle_{M_2}.$$

To prove that (1.16) indeed defines a Riemannian metric on $M_1 \times M_2$, it suffices to verify the following properties

(i) Since $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are bilinear,

$$\begin{aligned}\langle\lambda u + w, v\rangle &= \langle\lambda u_1 + w_1, v_1\rangle_1 + f(p)^2\langle\lambda u_2 + w_2, v_2\rangle_2\\ &= \lambda\langle u_1, v_1\rangle_1 + \langle w_1, v_1\rangle_1 + f(p)^2\lambda\langle u_2, v_2\rangle_2 + f(p)^2\langle w_2, v_2\rangle_2\\ &= \lambda\langle u, v\rangle + \langle w, v\rangle.\end{aligned}$$

(ii) Since $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are symmetric, we have

$$\begin{aligned}\langle u, v\rangle &= \langle u_1, v_1\rangle_1 + f(p)^2\langle u_2, v_2\rangle_2\\ &= \langle v_1, u_1\rangle_1 + f(p)^2\langle v_2, u_2\rangle_2\\ &= \langle v, u\rangle.\end{aligned}$$

(iii) Finally, in the case where $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are Riemannian metrics (that is, they satisfy the positive-definiteness property), we have

$$\langle u, u\rangle = \langle u_1, u_1\rangle_1 + f(p)^2\langle u_2, u_2\rangle_2 \geqslant 0$$

and

$$\langle u, u\rangle = 0 \iff u_1 = 0 \text{ and } u_2 = 0 \iff u = 0.$$

We will write $\overline{M} = M_1 \times_f M_2$ to denote the product manifold $M_1 \times M_2$ endowed with the metric (1.16), and in this case, we will say that $M_1 \times_f M_2$ is the **warped product** of M_1 by M_2 with the **warping function** f. The manifold M_1 is called the **base** and M_2 is called the **fiber** of the warped product $M_1 \times_f M_2$.

The submanifolds $M_1 \times \{q\}$ are called **leaves**, and $\{p\} \times M_2$ are the **fibers** (or, as we will see in the subsequent chapters, the **slices**). The following facts hold, the proofs of which follow immediately from the definition

(i) For each $q \in M_2$, the restriction $\pi_1|_{M_1\times\{q\}}$ is an isometry onto M_1;

(ii) For each $p \in M_1$, the restriction $\pi_2|_{\{p\}\times M_2}$ is a positive homothety onto M_2, that is, the metric on $\{p\} \times M_2$ is $f(p)^2\langle\cdot,\cdot\rangle_2$, with a scale factor of $\frac{1}{f(p)}$;

(iii) For each $(p, q) \in M_1 \times M_2$, the leaf $M_1 \times \{q\}$ and the slice $\{p\} \times M_2$ are orthogonal at (p, q).

We call the tangent vectors to the leaves *horizontal* and the tangent vectors to the fibers *vertical*. We are especially interested in computing the Levi-Civita connection ∇ and the curvature tensor R of a warped product $M_1 \times_f M_2$ from the knowledge of the connections ∇^1 and ∇^2 of M_1 and M_2, respectively.

First, we will compute the connection ∇ of M. Let us recall the Koszul formula

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\Big\{X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle\Big\}. \tag{1.17}$$

We will use the decomposition $X = X_1 + X_2$ of a vector field $X \in \mathfrak{X}(M)$ as a sum of vector fields tangent to M_1 and M_2, respectively. In the propositions and corollaries below, this decomposition will play a fundamental role.

Notice that

$$\begin{aligned} X\langle Y, Z\rangle &= (X_1 + X_2)\big(\langle Y_1, Z_1\rangle_1 + f^2\langle Y_2, Z_2\rangle_2\big) \\ &= X_1\langle Y_1, Z_1\rangle_1 + X_1[f^2]\langle Y_2, Z_2\rangle_2 + f^2 X_2\langle Y_2, Z_2\rangle_2, \end{aligned} \tag{1.18}$$

where $X_1[f^2]$ denotes the directional derivative of f^2 in the direction of X_1 (note that f does not depend on the second coordinate). Furthermore, by the properties of the Lie bracket, we have

$$\langle [X, Z], Y\rangle = \langle [X_1, Z_1], Y_1\rangle_1 + f^2\langle [X_2, Z_2], Y_2\rangle_2. \tag{1.19}$$

Therefore, applying analogous formulas to those in (1.18) and (1.19) for $Y\langle X, Z\rangle$, $Z\langle X, Y\rangle$, $\langle [Y, Z], X\rangle$, and $\langle [X, Y], Z\rangle$, and substituting them into (1.17), we obtain

$$\begin{aligned} \langle \nabla_X Y, Z\rangle &= \langle \nabla^1_{X_1} Y_1, Z_1\rangle_1 + f^2\langle \nabla^2_{X_2} Y_2, Z_2\rangle_2 \\ &\quad + f\langle X_1, \nabla^1 f\rangle_1\langle Y_2, Z_2\rangle_2 + f\langle Y_1, \nabla^1 f\rangle_1\langle X_2, Z_2\rangle_2 - f\langle Z_1, \nabla^1 f\rangle_1\langle X_2, Y_2\rangle_2 \\ &= \langle \nabla^1_{X_1} Y_1 - f\langle X_2, Y_2\rangle_2 \nabla^1 f, Z_1\rangle_1 \\ &\quad + f^2\left\langle \nabla^2_{X_2} Y_2 + \frac{\langle X_1, \nabla^1 f\rangle_1}{f} Y_2 + \frac{\langle Y_1, \nabla^1 f\rangle_1}{f} X_2, Z_2\right\rangle_2. \end{aligned}$$

Consequently, since Z is arbitrary, we can conclude from the expression above that

$$\nabla_X Y = \nabla^1_{X_1} Y_1 - f\langle X_2, Y_2\rangle_2 \nabla^1 f + \nabla^2_{X_2} Y_2 + \frac{\langle X_1, \nabla^1 f\rangle_1}{f} Y_2 + \frac{\langle Y_1, \nabla^1 f\rangle_1}{f} X_2. \tag{1.20}$$

The following proposition outlines the behavior of the Levi-Civita connection within the framework of the warped product manifold $M_1 \times_f M_2$.

Proposition 1.4.1. *Let* $X_1, Y_1 \in \mathfrak{X}(M_1)$ *and* $X_2, Y_2 \in \mathfrak{X}(M_2)$. *Let* ∇, ∇^1, *and* ∇^2 *be the Levi-Civita connections of* $M_1 \times_f M_2$, M_1, *and* M_2, *respectively. Then*

(i) $\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1$.

(ii) $\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = \frac{X_1[f]}{f} X_2$.

(iii) $\nabla_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f\langle X_2, Y_2\rangle_2 \nabla^1 f$.

We now proceed to compute the Riemann curvature tensor R of $M_1 \times_f M_2$ in terms of the curvature tensors R_1 and R_2 of M_1 and M_2, respectively. As previously established, we shall adopt the following convention for the Riemann curvature tensor

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z. \tag{1.21}$$

Where $[\cdot,\cdot]$ denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

The properties of the Riemann curvature tensor on $M_1 \times_f M_2$ outlined below will be crucial for our subsequent results.

Proposition 1.4.2. *Let* $X_1, Y_1, Z_1 \in \mathfrak{X}(M_1)$ *and* $X_2, Y_2, Z_2 \in \mathfrak{X}(M_2)$. *Let* $\nabla, \nabla^1, \nabla^2$ *and* R, R^1, R^2 *be the Levi-Civita connections and Riemann curvature tensors of* $M_1 \times_f M_2$, M_1, *and* M_2, *respectively. Then*

(i) $R(X_1, Y_1)Z_1 = R^1(X_1, Y_1)Z_1$

(ii) $R(X_1, Y_1)Z_2 = 0$

(iii) $R(X_1, Y_2)Z_1 = -\frac{\mathrm{Hess}^1 f(X_1, Z_1)}{f} Y_2$

(iv) $R(X_1, Y_2)Z_2 = f\langle Y_2, Z_2\rangle_2 \nabla^1_{X_1}(\nabla^1 f) = \frac{\langle Y_2, Z_2\rangle}{f} \nabla^1_{X_1}(\nabla^1 f)$.

(v) $R(X_2, Y_2)Z_1 = 0$.

(vi) $R(X_2, Y_2)Z_2 = R^2(X_2, Y_2)Z_2 - \langle \nabla^1 f, \nabla^1 f\rangle_1 \big\{\langle X_2, Z_2\rangle_2 Y_2 - \langle Y_2, Z_2\rangle_2 X_2\big\}$,

where $\mathrm{Hess}_1 f$ *denotes the Hessian of* f *with respect to the metric* $\langle \cdot,\cdot\rangle_1$.

Proof. We compute each item directly using the geometric identities established in Proposition 1.4.1. Throughout the proof, recall that $[X_1, X_2] = 0$ for any horizontal field $X_1 \in \mathfrak{X}(M_1)$ and vertical field $X_2 \in \mathfrak{X}(M_2)$.

(i) It follows immediately by **Proposition 1.4.1 (i)**

$$\begin{aligned} R(X_1, Y_1)Z_1 &= \nabla_{Y_1}\nabla_{X_1} Z_1 - \nabla_{X_1}\nabla_{Y_1} Z_1 - \nabla_{[X_1,Y_1]} Z_1 \\ &= \nabla^1_{Y_1}\nabla^1_{X_1} Z_1 - \nabla^1_{X_1}\nabla^1_{Y_1} Z_1 - \nabla^1_{[X_1,Y_1]} Z_1 \\ &= R^1(X_1, Y_1)Z_1. \end{aligned}$$

(ii) By the **Proposition 1.4.1 (ii)**, yields

$$
\begin{aligned}
\mathsf{R}(\mathsf{X}_1,\mathsf{Y}_1)\mathsf{Z}_2 &= \nabla_{\mathsf{Y}_1}\nabla_{\mathsf{X}_1}\mathsf{Z}_2 - \nabla_{\mathsf{X}_1}\nabla_{\mathsf{Y}_1}\mathsf{Z}_2 - \nabla_{[\mathsf{X}_1,\mathsf{Y}_1]}\mathsf{Z}_2\\
&= \nabla_{\mathsf{Y}_1}\left(\frac{\langle \mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\mathsf{Z}_2\right) - \nabla_{\mathsf{X}_1}\left(\frac{\langle \mathsf{Y}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\mathsf{Z}_2\right) - \frac{\langle [\mathsf{X}_1,\mathsf{Y}_1],\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\mathsf{Z}_2\\
&= \frac{\langle \mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\frac{\langle \mathsf{Y}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\mathsf{Z}_2 + \mathsf{Y}_1\left(\frac{\langle \mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\right)\mathsf{Z}_2\\
&\quad - \frac{\langle \mathsf{Y}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\frac{\langle \mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\mathsf{Z}_2 - \mathsf{X}_1\left(\frac{\langle \mathsf{Y}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\right)\mathsf{Z}_2\\
&\quad - \frac{\langle \nabla^1_{\mathsf{X}_1}\mathsf{Y}_1 - \nabla^1_{\mathsf{Y}_1}\mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\mathsf{Z}_2\\
&= \left(\frac{\mathsf{f}\langle \nabla^1_{\mathsf{Y}_1}\mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1 + \langle \mathsf{X}_1,\nabla^1_{\mathsf{Y}_1}\nabla^1\mathsf{f}\rangle_1 - \langle \mathsf{Y}_1,\nabla^1\mathsf{f}\rangle_1\langle \mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}^2}\right)\mathsf{Z}_2\\
&\quad - \left(\frac{\mathsf{f}\langle \nabla^1_{\mathsf{X}_1}\mathsf{Y}_1,\nabla^1\mathsf{f}\rangle_1 + \langle \mathsf{Y}_1,\nabla^1_{\mathsf{X}_1}\nabla^1\mathsf{f}\rangle_1 - \langle \mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1\langle \mathsf{Y}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}^2}\right)\mathsf{Z}_2\\
&\quad - \frac{\langle \nabla^1_{\mathsf{X}_1}\mathsf{Y}_1 - \nabla^1_{\mathsf{Y}_1}\mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\mathsf{Z}_2\\
&= \frac{(\mathrm{Hess}^1\,\mathsf{f})(\mathsf{X}_1,\mathsf{Y}_1)}{\mathsf{f}}\mathsf{Z}_2 - \frac{(\mathrm{Hess}^1\,\mathsf{f})(\mathsf{X}_1,\mathsf{Y}_1)}{\mathsf{f}}\mathsf{Z}_2 = 0.
\end{aligned}
$$

(iii) Since $[\mathsf{X}_1,\mathsf{Y}_2] = 0$, the definition simplifies to

$$
\begin{aligned}
\mathsf{R}(\mathsf{X}_1,\mathsf{Y}_2)\mathsf{Z}_1 &= \nabla_{\mathsf{Y}_2}\nabla_{\mathsf{X}_1}\mathsf{Z}_1 - \nabla_{\mathsf{X}_1}\nabla_{\mathsf{Y}_2}\mathsf{Z}_1 - \nabla_{[\mathsf{X}_1,\mathsf{Y}_2]}\mathsf{Z}_1\\
&= \nabla_{\mathsf{Y}_2}(\nabla^1_{\mathsf{X}_1}\mathsf{Z}_1) - \nabla_{\mathsf{X}_1}\left(\frac{\langle \mathsf{Z}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\mathsf{Y}_2\right)\\
&= \frac{\langle \nabla^1_{\mathsf{X}_1}\mathsf{Z}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\mathsf{Y}_2 - \mathsf{X}_1\left(\frac{\langle \mathsf{Z}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\right)\mathsf{Y}_2 - \frac{\langle \mathsf{Z}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\frac{\langle \mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\mathsf{Y}_2.
\end{aligned}
$$

But note that

$$
\mathsf{X}_1\left(\frac{\langle \mathsf{Z}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\right)\mathsf{Y}_2 = \left(\frac{\mathsf{f}\langle \nabla^1_{\mathsf{X}_1}\mathsf{Z}_1,\nabla^1\mathsf{f}\rangle_1 + \langle \mathsf{Z}_1,\nabla^1_{\mathsf{X}_1}\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}^2}\right)\mathsf{Y}_2 - \frac{\langle \mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1\langle \mathsf{Z}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}^2}\mathsf{Y}_2.
$$

Therefore, performing the cancellations, we obtain

$$
\mathsf{R}(\mathsf{X}_1,\mathsf{Y}_2)\mathsf{Z}_1 = -\frac{(\mathrm{Hess}^1\,\mathsf{f})(\mathsf{X}_1,\mathsf{Z}_1)}{\mathsf{f}}\mathsf{Y}_2.
$$

(iv) Using the connection properties for mixed and vertical fields, we expand the expression as

$$
\begin{aligned}
\mathsf{R}(\mathsf{X}_1,\mathsf{Y}_2)\mathsf{Z}_2 &= \nabla_{\mathsf{Y}_2}\nabla_{\mathsf{X}_1}\mathsf{Z}_2 - \nabla_{\mathsf{X}_1}\nabla_{\mathsf{Y}_2}\mathsf{Z}_2 - \nabla_{[\mathsf{X}_1,\mathsf{Y}_2]}\mathsf{Z}_2\\
&= \nabla_{\mathsf{Y}_2}\left(\frac{\langle \mathsf{X}_1,\nabla^1\mathsf{f}\rangle_1}{\mathsf{f}}\mathsf{Z}_2\right) - \nabla_{\mathsf{X}_1}\left(\nabla^2_{\mathsf{Y}_2}\mathsf{Z}_2 - \mathsf{f}\langle \mathsf{Y}_2,\mathsf{Z}_2\rangle_2\nabla^1\mathsf{f}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\langle X_1, \nabla^1 f\rangle_1}{f}\nabla^2_{Y_2} Z_2 - f\langle Y_2, Z_2\rangle_2 \nabla^1 f + Y_2\left(\frac{\langle X_1, \nabla^1 f\rangle_1}{f}\right) Z_2 \\
&\quad - \frac{\langle X_1, \nabla^1 f\rangle_1}{f}\nabla^2_{Y_2} Z_2 + \langle X_1, \nabla^1 f\rangle_1 \langle Y_2, Z_2\rangle_2 \nabla^1 f + f\langle Y_2, Z_2\rangle_2 \nabla^1_{X_1}\nabla^1 f \\
&= f\langle Y_2, Z_2\rangle_2 \nabla^1_{X_1}\nabla^1 f.
\end{aligned}
$$

(v) Applying the mixed connection formulas to each term, we have

$$
\begin{aligned}
R(X_2, Y_2)Z_1 &= \nabla_{Y_2}\nabla_{X_2} Z_1 - \nabla_{X_2}\nabla_{Y_2} Z_1 - \nabla_{[X_2,Y_2]} Z_1 \\
&= \nabla_{Y_2}\left(\frac{\langle Z_1, \nabla^1 f\rangle_1}{f} X_2\right) - \nabla_{X_2}\left(\frac{\langle Z_1, \nabla^1 f\rangle_1}{f} Y_2\right) + \frac{\langle Z_1, \nabla^1 f\rangle_1}{f}[X_2, Y_2] \\
&= \frac{\langle Z_1, \nabla^1 f\rangle_1}{f}\nabla^2_{Y_2} X_2 - f\langle Y_2, X_2\rangle_2 \nabla^1 f + Y_2\left(\frac{\langle Z_1, \nabla^1 f\rangle_1}{f}\right) X_2 \\
&\quad - \frac{\langle Z_1, \nabla^1 f\rangle_1}{f}\nabla^2_{X_2} Y_2 - f\langle X_2, Y_2\rangle_2 \nabla^1 f + X_2\left(\frac{\langle Z_1, \nabla^1 f\rangle_1}{f}\right) Y_2 \\
&\quad + \frac{\langle Z_1, \nabla^1 f\rangle_1}{f}\nabla^2_{X_2} Y_2 - \nabla^2_{Y_2} X_2 = 0.
\end{aligned}
$$

(vi) Substituting the vertical connection formulas into the definition, we get

$$
\begin{aligned}
R(X_2, Y_2)Z_2 &= \nabla_{Y_2}\nabla_{X_2} Z_2 - \nabla_{X_2}\nabla_{Y_2} Z_2 - \nabla_{[X_2,Y_2]} Z_2 \\
&= \nabla_{Y_2}\Big(\nabla^2_{X_2} Z_2 - f\langle X_2, Z_2\rangle_2 \nabla^1 f\Big) - \nabla_{X_2}\Big(\nabla^2_{Y_2} Z_2 - f\langle Y_2, Z_2\rangle_2 \nabla^1 f\Big) \\
&\quad + \nabla^2_{[X_2,Y_2]} Z_2 - f\langle [X_2, Y_2], Z_2\rangle_2 \nabla^1 f \\
&= R^2(X_2, Y_2)Z_2 - f\langle Y_2, \nabla^2_{X_2} Z_2\rangle_2 \nabla^1 f + f\langle X_2, \nabla^2_{Y_2} Z_2\rangle_2 \nabla^1 f \\
&\quad - f\langle X_2, Z_2\rangle_2 \frac{\langle \nabla^1 f, \nabla^1 f\rangle_1}{f} Y_2 - f\Big(\langle \nabla^2_{Y_2} X_2, Z_2\rangle_2 + \langle X_2, \nabla^2_{Y_2} Z_2\rangle_2\Big)\nabla^1 f \\
&\quad + f\langle Y_2, Z_2\rangle_2 \frac{\langle \nabla^1 f, \nabla^1 f\rangle_1}{f} X_2 + f\Big(\langle \nabla^2_{X_2} Y_2, Z_2\rangle_2 + \langle Y_2, \nabla^2_{X_2} Z_2\rangle_2\Big)\nabla^1 f \\
&\quad - f\langle \nabla^2_{X_2} Y_2 - \nabla^2_{Y_2} X_2, Z_2\rangle_2 \nabla^1 f \\
&= R^2(X_2, Y_2)Z_2 - \langle \nabla^1 f, \nabla^1 f\rangle_1 \Big(\langle X_2, Z_2\rangle_2 Y_2 - \langle Y_2, Z_2\rangle_2 X_2\Big) \\
&= R^2(X_2, Y_2)Z_2 - \frac{\langle \nabla^1 f, \nabla^1 f\rangle_1}{f^2}\Big(\langle X_2, Z_2\rangle Y_2 - \langle Y_2, Z_2\rangle X_2\Big).
\end{aligned}
$$

which completes the proof of the proposition. □

As an immediate consequence of the previous identities, we obtain the following relations for the components of the Ricci tensor.

Proposition 1.4.3. *In the warped product* $M_1 \times_f M_2$ *with* $\dim(M_2) = m > 1$*, let* $X_1, Y_1 \in \mathfrak{X}(M_1)$ *and* $X_2, Y_2 \in \mathfrak{X}(M_2)$*. If* Ric*,* Ric_1*, and* Ric_2 *denote the Ricci tensors of* $M_1 \times_f M_2$*,* M_1*, and* M_2*, respectively, then*

(i) $\mathrm{Ric}(X_1, Y_1) = \mathrm{Ric}^1(X_1, Y_1) - \frac{m}{f}(\mathrm{Hess}^1 f)(X_1, Y_1)$,

(ii) $\mathrm{Ric}(X_1, Y_2) = 0$,

(iii) $\mathrm{Ric}(X_2, Y_2) = \mathrm{Ric}^2(X_2, Y_2) - \frac{\langle X_2, Y_2\rangle_2}{f}\Delta_1 f + \frac{m-1}{f^2}\langle \nabla^1 f, \nabla^1 f\rangle_1 \langle X_2, Y_2\rangle_2$,

where $\Delta_1 f$ *and* $\nabla^1 f$ *denote the Laplacian and the gradient of* f *on* M_1*, respectively.*

Proof. Let $n = \dim(M_1)$ and $m = \dim(M_2)$. Let $\{E_i\}_{i=1}^n \cup \{F_j\}_{j=1}^m$ be an orthonormal frame on an open subset of $M_1 \times_f M_2$ such that the fields $\{E_i\}$ are tangent to M_1 and $\{F_j\}$ are tangent to M_2.

(i) For the purely horizontal component, we compute $\mathrm{Ric}(X_1, Y_1)$ by decomposing the trace into its base and fiber directions

$$\begin{aligned}
\mathrm{Ric}(X_1, Y_1) &= \sum_{i=1}^{n} \langle R(X_1, E_i)Y_1, E_i\rangle + \sum_{j=1}^{m} \langle R(X_1, F_j)Y_1, F_j\rangle \\
&= \sum_{i=1}^{n} \langle R^1(X_1, E_i)Y_1, E_i\rangle_1 + \sum_{j=1}^{m} \left\langle -\frac{1}{f}(\mathrm{Hess}^1 f)(X_1, Y_1)F_j, F_j \right\rangle \\
&= \mathrm{Ric}^1(X_1, Y_1) - \frac{1}{f}(\mathrm{Hess}^1 f)(X_1, Y_1) \sum_{j=1}^{m} \langle F_j, F_j\rangle \\
&= \mathrm{Ric}^1(X_1, Y_1) - \frac{1}{f}(\mathrm{Hess}^1 f)(X_1, Y_1) \cdot m \\
&= \mathrm{Ric}^1(X_1, Y_1) - \frac{m}{f}(\mathrm{Hess}^1 f)(X_1, Y_1).
\end{aligned}$$

(ii) For the mixed terms where $X_1 \in \mathfrak{X}(M_1)$ and $Y_2 \in \mathfrak{X}(M_2)$, evaluating the trace of the curvature tensor yields

$$\begin{aligned}
\mathrm{Ric}(X_1, Y_2) &= \sum_{i=1}^{n} \langle R(X_1, E_i)Y_2, E_i\rangle + \sum_{j=1}^{m} \langle R(X_1, F_j)Y_2, F_j\rangle \\
&= 0 + \sum_{j=1}^{m} \left\langle \frac{1}{f}\langle F_j, Y_2\rangle \nabla^1_{X_1}\nabla^1 f, F_j \right\rangle \\
&= \frac{1}{f}\sum_{j=1}^{m} \langle F_j, Y_2\rangle \cdot \langle \nabla^1_{X_1}\nabla^1 f, F_j\rangle \\
&= \frac{1}{f}\sum_{j=1}^{m} \langle F_j, Y_2\rangle \cdot 0 \\
&= 0.
\end{aligned}$$

(iii) By a similar application of our previous results, it follows that

$$\begin{aligned}
\mathrm{Ric}(X_2, Y_2) &= \sum_{i=1}^{n} \langle R(X_2, E_i)Y_2, E_i\rangle + \sum_{j=1}^{m} \langle R(X_2, F_j)Y_2, F_j\rangle \\
&= \sum_{i=1}^{n} \left\langle -\frac{1}{f}\langle X_2, Y_2\rangle \nabla^1_{E_i}\nabla^1 f, E_i \right\rangle \\
&\quad + \sum_{j=1}^{m} \left\langle R^2(X_2, F_j)Y_2 - \frac{\langle \nabla^1 f, \nabla^1 f\rangle_1}{f^2} \left(\langle F_j, Y_2\rangle X_2 - \langle X_2, Y_2\rangle F_j\right), F_j \right\rangle \\
&= -\frac{\langle X_2, Y_2\rangle}{f} \sum_{i=1}^{n} \langle \nabla^1_{E_i}\nabla^1 f, E_i\rangle_1 + \sum_{j=1}^{m} \langle R^2(X_2, F_j)Y_2, F_j\rangle \\
&\quad - \frac{\langle \nabla^1 f, \nabla^1 f\rangle_1}{f^2} \sum_{j=1}^{m} \left[\langle F_j, Y_2\rangle\langle X_2, F_j\rangle - \langle X_2, Y_2\rangle\langle F_j, F_j\rangle\right] \\
&= -\frac{f^2\langle X_2, Y_2\rangle_2}{f}\Delta_1 f + \mathrm{Ric}^2(X_2, Y_2) \\
&\quad - \frac{\langle \nabla^1 f, \nabla^1 f\rangle_1}{f^2} \left[f^2\langle X_2, Y_2\rangle_2 - \langle X_2, Y_2\rangle \cdot m\right] \\
&= -f\langle X_2, Y_2\rangle_2 \Delta_1 f + \mathrm{Ric}_2(X_2, Y_2) - \frac{\langle \nabla^1 f, \nabla^1 f\rangle_1}{f^2} \left[f^2\langle X_2, Y_2\rangle_2 - mf^2\langle X_2, Y_2\rangle_2\right] \\
&= \mathrm{Ric}_2(X_2, Y_2) - f\langle X_2, Y_2\rangle_2 \Delta_1 f - \frac{\langle \nabla^1 f, \nabla^1 f\rangle_1}{f^2} \cdot f^2\langle X_2, Y_2\rangle_2(1 - m) \\
&= \mathrm{Ric}_2(X_2, Y_2) - \frac{\langle X_2, Y_2\rangle}{f}\Delta_1 f + \frac{m-1}{f^2}\langle \nabla^1 f, \nabla^1 f\rangle_1 \langle X_2, Y_2\rangle.
\end{aligned}$$

□

The following result is key to unifying the geometric concepts that link the intrinsic curvature of the factor manifolds to the global warped structure.

Corollary 1.4.1. *The warped product* $M_1 \times_f M_2$*, with* $\dim(M_2) = m > 1$*, is Einstein with* $\mathrm{Ric} = \lambda g$ *if and only if*

(i) $\mathrm{Ric}^1(X_1, Y_1) = \lambda g_1(X_1, Y_1) + \frac{m}{f}(\mathrm{Hess}^1 f)(X_1, Y_1)$,

(ii) (M_2, g_2) *is Einstein with* $\mathrm{Ric}^2(X_2, Y_2) = \mu g_2(X_2, Y_2)$,

(iii) $-f\Delta_1 f + (m-1)\langle \nabla^1 f, \nabla^1 f\rangle_1 + \lambda f^2 = \mu$,

for all $X_1, Y_1 \in \mathfrak{X}(M_1)$ *and* $X_2, Y_2 \in \mathfrak{X}(M_2)$.

Chapter 2

Examples and Properties

This chapter is dedicated to the study of quasi-Einstein manifolds, focusing on their fundamental properties and exploring concrete examples that illustrate their structural and geometric behavior.

2.1 Quasi-Einstein manifolds

To properly contextualize these examples, we must first introduce the notion and basic structural features of quasi-Einstein manifolds.

Definition 2.1.1. *A complete Riemannian manifold* (M^n, g)*,* $n \geqslant 2$*, possibly with boundary, is called an* m-quasi-Einstein manifold*, or simply a* quasi-Einstein manifold*, if there exists a smooth potential function* u *on* M *satisfying the fundamental equation*

$$\mathsf{Hess}\, u = \frac{u}{m} (\mathrm{Ric} - \lambda g), \tag{1.5}$$

for some constants λ *and* m*. Moreover,* $u > 0$ *in* $\mathrm{int}(M)$*, and* $u = 0$ *on* ∂M*. Here,* $\mathsf{Hess}\, u$ *and* Ric *denote the Hessian of* u *and the Ricci curvature of* M*, respectively.*

The study of these metrics is intrinsically connected to the geometry of Einstein warped products. Indeed, the Corollary 1.4.1 allow us to conclude that when m is a positive integer, a quasi-Einstein metric is precisely the base metric of an $(n+m)$-dimensional Einstein warped product space. More specifically, the warped product $(M \times F^m, g + e^{-\frac{2f}{m}} g_F)$ is an Einstein manifold with Einstein constant λ, where (F^m, g_F) is an m-dimensional Einstein manifold. Furthermore, under the substitution $u = e^{-\frac{f}{m}}$, a straightforward computation yields the localized quasi-Einstein equation (1). In fact , by calculating the derivatives of u

$$\nabla_j u = -\frac{1}{m} e^{-f/m} \nabla_j f = -\frac{u}{m} \nabla_j f.$$

A second covariant differentiation yields the components of the Hessian tensor. Applying the product rule, yields

$$\begin{aligned}
\nabla_i \nabla_j u &= \nabla_i \left(-\frac{u}{m} \nabla_j f \right) \\
&= -\frac{1}{m} (\nabla_i u) \nabla_j f - \frac{u}{m} \nabla_i \nabla_j f \\
&= -\frac{1}{m} \left(-\frac{u}{m} \nabla_i f \right) \nabla_j f - \frac{u}{m} \nabla_i \nabla_j f \\
&= \frac{u}{m^2} \nabla_i f \nabla_j f - \frac{u}{m} \nabla_i \nabla_j f.
\end{aligned}$$

We now equate this expression with the right-hand side of the structural condition (2.1.1)

$$-\frac{u}{m} \nabla_i \nabla_j f + \frac{u}{m^2} \nabla_i f \nabla_j f = \frac{u}{m} (R_{ij} - \lambda g_{ij}).$$

Since $u > 0$ on the interior of M, we can divide the entire identity by $\frac{u}{m}$, which gives

$$-\nabla_i \nabla_j f + \frac{1}{m} \nabla_i f \nabla_j f = R_{ij} - \lambda g_{ij}.$$

Rearranging the terms immediately produces the m-Bakry-Emery Ricci identity in local coordinates

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{m} \nabla_i f \nabla_j f = \lambda g_{ij}. \tag{2.1}$$

This analysis directly extends the study and results found in [16].

Below, we present the foundational examples of quasi-Einstein manifolds that appear in the literature.

2.2 Standard hemisphere

This example provides a concrete verification of the standard hemisphere, establishing its classification as a compact m-quasi-Einstein manifold.

Example 1. *Consider the standard hemisphere $\mathbb{S}^n_+$ equipped with the metric*

$$g = dr^2 + \sin^2 r \, g_{\mathbb{S}^{n-1}},$$

and the potential function $u(r) = \cos r$*, for* $r \leqslant \pi/2$*. Setting* $\lambda = m+n-1$ *for an arbitrary finite constant* m*,* $\mathbb{S}^n_+$ *becomes a compact, oriented* m*-quasi-Einstein manifold with boundary* $\mathbb{S}^{n-1}$*. This is verified by noting that* $u = \cos(\pi/2) = 0$*, along the boundary and* $u > 0$ *throughout the interior.*

Using the identity $\nabla_i \nabla_j r = \sin r \cos r\,(g_{\mathbb{S}^{n-1}})_{ij}$*, the Hessian of the potential function expands via the chain rule as follows*

$$\begin{aligned}\nabla_i \nabla_j u &= -\cos r\, dr^2 - \sin r\, \nabla_i \nabla_j r \\ &= -\cos r\, dr^2 - \sin^2 r \cos r\,(g_{\mathbb{S}^{n-1}})_{ij} \\ &= -\cos r \left(dr^2 + \sin^2 r\,(g_{\mathbb{S}^{n-1}})_{ij}\right) \\ &= -u g_{ij}.\end{aligned}$$

In parallel, since the round metric satisfies $R_{ij} = (n-1)g_{ij}$*, substitution into the structural expression gives*

$$\begin{aligned}\frac{u}{m}(R_{ij} - \lambda g_{ij}) &= \frac{u}{m}\big((n-1)g_{ij} - (m+n-1)g_{ij}\big) \\ &= \frac{u}{m}(-m g_{ij}) \\ &= -u g_{ij}.\end{aligned}$$

Comparing both results yields $\nabla_i \nabla_j u = \frac{u}{m}(R_{ij} - \lambda g_{ij})$*, establishing that it constitutes a quasi-Einstein manifold.*

2.3 Cylinder

Another classic example of an quasi-Einstein manifold is the Riemannian cylinder, which we verify below.

Example 2. *Let* $\lambda > 0$ *be a real constant and consider the manifold* $M = I \times \mathbb{S}^{n-1}$ *over the interval* $I = \left[0, \frac{\sqrt{m}}{\sqrt{\lambda}}\pi\right]$*. We equip* M *with the product Riemannian metric*

$$g = dt^2 + \frac{n-2}{\lambda} g_{\mathbb{S}^{n-1}}$$

and define the potential function by $u(t) = \sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)$*. Under this choice of data,* M *becomes a compact, oriented* m*-quasi-Einstein manifold with a disconnected boundary consisting of two disjoint copies of the sphere*

$$\partial M = \left(\{0\} \times \mathbb{S}^{n-1}\right) \cup \left(\left\{\frac{\sqrt{m}}{\sqrt{\lambda}}\pi\right\} \times \mathbb{S}^{n-1}\right) = \partial M_1 \cup \partial M_2.$$

To verify the boundary behavior, we notice that $u = \sin(0) = 0$ *on* ∂M_1*, while* $u = \sin(\pi) = 0$ *on* ∂M_2*, ensuring that* $u = 0$ *identically on* ∂M*. Furthermore,* $u > 0$ *holds throughout the interior* $\mathrm{int}(M)$*.*

To check the structural equation in index notation, we recall that the slice metric has a constant warping factor in t*. Consequently, the Hessian of the distance coordinate vanishes identically,* $\nabla_i \nabla_j t = 0$*. Applying the standard chain rule to the potential function* $u(t)$ *yields*

$$\begin{aligned} \nabla_j u &= \nabla_j \left(\sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}} t \right) \right) \\ &= \frac{\sqrt{\lambda}}{\sqrt{m}} \cos\left(\frac{\sqrt{\lambda}}{\sqrt{m}} t \right) \nabla_j t, \end{aligned}$$

and then, the Hessian components $\nabla_j \nabla_i u$ *expand as*

$$\begin{aligned} \nabla_i \nabla_j u &= \nabla_i \left(\frac{\sqrt{\lambda}}{\sqrt{m}} \cos\left(\frac{\sqrt{\lambda}}{\sqrt{m}} t \right) \nabla_j t \right) \\ &= \frac{\sqrt{\lambda}}{\sqrt{m}} \cos\left(\frac{\sqrt{\lambda}}{\sqrt{m}} t \right) \nabla_i \nabla_j t - \frac{\lambda}{m} \sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}} t \right) dt^2 \\ &= 0 - \frac{\lambda}{m} u \, dt^2 \\ &= -\frac{\lambda}{m} u \, dt^2. \end{aligned}$$

In parallel, we evaluate the right-hand side of the structural profile. Since the ambient space is a direct product, the Ricci tensor has no radial components and is determined entirely by the spherical factor $R_{ij} = (n-2)(g_{\mathbb{S}^{n-1}})_{ij}$*. Decomposing the full metric as* $g_{ij} = dt^2 + \frac{n-2}{\lambda}(g_{\mathbb{S}^{n-1}})_{ij}$*, we compute*

$$\begin{aligned} \frac{u}{m} (R_{ij} - \lambda g_{ij}) &= \frac{u}{m} \left[(n-2)(g_{\mathbb{S}^{n-1}})_{ij} - \lambda \left(dt^2 + \frac{n-2}{\lambda} (g_{\mathbb{S}^{n-1}})_{ij} \right) \right] \\ &= \frac{u}{m} \left[(n-2)(g_{\mathbb{S}^{n-1}})_{ij} - \lambda dt^2 - (n-2)(g_{\mathbb{S}^{n-1}})_{ij} \right] \\ &= \frac{u}{m} \left(-\lambda dt^2 \right) \\ &= -\frac{\lambda}{m} u \, dt^2. \end{aligned}$$

Comparing both explicit expansions, we find $\nabla_i \nabla_j u = \frac{u}{m}(R_{ij} - \lambda g_{ij})$*, completing the verification.*

2.4 Product of spheres

We conclude the examples by turning our attention to product spheres. As a prerequisite for this construction, it is crucial to analyze how the curvature tensor and the Ricci tensor behave under a general doubly warped product configuration.

To compute the Riemann curvature profile of the general doubly warped product metric

$$g = dr^2 + \phi(r)^2 g_{\mathbb{S}^p} + \psi(r)^2 g_{\mathbb{S}^q}$$

on $M = I \times \mathbb{S}^p \times \mathbb{S}^q$, we construct an adapted orthonormal frame. Let $\{E_1, \dots, E_p\}$ be a local orthonormal frame on the standard unit sphere $(\mathbb{S}^p, g_{\mathbb{S}^p})$, and let $\{E_{p+1}, \dots, E_{p+q}\}$ be a local orthonormal frame on $(\mathbb{S}^q, g_{\mathbb{S}^q})$. Together, they form an orthonormal basis for the product manifold $(\mathbb{S}^p \times \mathbb{S}^q, g_{\mathbb{S}^p} + g_{\mathbb{S}^q})$.

Recalling these vectors to account for the warping functions, we define our local ambient orthonormal frame $\{F_0, F_1, \dots, F_{p+q}\}$ on (M, g) by

$$F_0 = \partial_r, \quad F_i = \frac{1}{\phi(r)} E_i \quad (1 \leqslant i \leqslant p), \quad F_\alpha = \frac{1}{\psi(r)} E_\alpha \quad (p+1 \leqslant \alpha \leqslant p+q).$$

Using the fact that ∂_r commutes with the coordinate vector fields of the fibers, the only non-vanishing Lie brackets involving the radial direction are

$$\begin{aligned} [F_0, F_i] &= \left[\partial_r, \frac{1}{\phi} E_i\right] \\ &= \partial_r\left(\phi^{-1}\right) E_i + \frac{1}{\phi}[\partial_r, E_i] \\ &= -\frac{\phi'}{\phi^2} E_i + 0 \\ &= -\frac{\phi'}{\phi} F_i, \end{aligned}$$

and

$$[F_0, F_\alpha] = -\frac{\psi'}{\psi} F_\alpha.$$

By evaluating Koszul's formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y),$$

we find the non-zero covariant derivatives involving the radial frame vector field

$$\begin{aligned} \nabla_{F_i} F_0 &= \frac{\phi'}{\phi} F_i, & \nabla_{F_\alpha} F_0 &= \frac{\psi'}{\psi} F_\alpha, \\ \nabla_{F_i} F_j &= \frac{1}{\phi}(\nabla^{\mathbb{S}^p}_{E_i} E_j) - \frac{\phi'}{\phi}\delta_{ij} F_0, & \nabla_{F_\alpha} F_\beta &= \frac{1}{\psi}(\nabla^{\mathbb{S}^q}_{E_\alpha} E_\beta) - \frac{\psi'}{\psi}\delta_{\alpha\beta} F_0. \end{aligned}$$

Crucially, because the spheres form a direct product topology, the mixed Lie brackets vanish ($[E_i, E_\alpha] = 0$), which implies the complete decoupling of the fiber connections

$$\nabla_{F_i} F_\alpha = 0, \quad \nabla_{F_\alpha} F_i = 0 \quad \text{for all } 1 \leqslant i \leqslant p < \alpha \leqslant p+q.$$

Using the standard algebraic definition of the curvature $\mathfrak{R}(F_A \wedge F_B) = g(R(F_A, F_B)F_B, F_A)$, we compute the components block-by-block.

Evaluating the curvature tensor yields

$$\begin{aligned} R(F_0, F_i)F_i &= \nabla_{F_0}\nabla_{F_i}F_i - \nabla_{F_i}\nabla_{F_0}F_i - \nabla_{[F_0,F_i]}F_i \\ &= \nabla_{F_0}\left(-\frac{\phi'}{\phi}F_0\right) - \nabla_{F_i}(0) - \left(-\frac{\phi'}{\phi}\right)\left(-\frac{\phi'}{\phi}F_0\right) \\ &= -\left(\frac{\phi''\phi - (\phi')^2}{\phi^2}\right)F_0 - \frac{(\phi')^2}{\phi^2}F_0 = -\frac{\phi''}{\phi}F_0. \end{aligned}$$

Taking the inner product with F_0 gives the standard radial sectional curvatures

$$\mathfrak{R}(F_0 \wedge F_i) = -\frac{\phi''(r)}{\phi(r)} \quad (1 \leqslant i \leqslant p), \quad \mathfrak{R}(F_0 \wedge F_\alpha) = -\frac{\psi''(r)}{\psi(r)} \quad (p+1 \leqslant \alpha \leqslant p+q).$$

For two distinct vectors tangent to the same spherical factor, the intrinsic round curvature of the fibers interacts with the radial acceleration. For the $\mathbb{S}^p$ factor

$$\mathfrak{R}(F_i \wedge F_j) = \frac{1}{\phi^2}\mathfrak{R}_{\mathbb{S}^p}(E_i \wedge E_j) - \frac{(\phi')^2}{\phi^2}.$$

Since the standard sphere has constant sectional curvature $\mathfrak{R}_{\mathbb{S}^p} = 1$, we obtain

$$\begin{aligned} \mathfrak{R}(F_i \wedge F_j) &= \frac{1 - (\phi'(r))^2}{\phi(r)^2} \quad (1 \leqslant i < j \leqslant p), \quad \mathfrak{R}(F_\alpha \wedge F_\beta) \\ &= \frac{1 - (\psi'(r))^2}{\psi(r)^2} \quad (p+1 \leqslant \alpha < \beta \leqslant p+q). \end{aligned}$$

Finally, we compute the curvature of planes spanned by one vector from each fiber. Because $\nabla_{F_i} F_\alpha = 0$, the curvature expression simplifies dramatically

$$\begin{aligned} R(F_i, F_\alpha)F_\alpha &= \nabla_{F_i}\nabla_{F_\alpha}F_\alpha - \nabla_{F_\alpha}\nabla_{F_i}F_\alpha - \nabla_{[F_i,F_\alpha]}F_\alpha \\ &= \nabla_{F_i}\left(-\frac{\psi'}{\psi}F_0\right) - 0 - 0 \\ &= -\frac{\psi'}{\psi}\left(\frac{\phi'}{\phi}F_i\right) = -\frac{\phi'\psi'}{\phi\psi}F_i. \end{aligned}$$

Taking the inner product with F_i reveals the cross-fiber sectional curvature

$$\mathfrak{R}(F_i \wedge F_\alpha) = -\frac{\phi'(r)\psi'(r)}{\phi(r)\psi(r)} \quad \text{for } 1 \leqslant i \leqslant p \text{ and } p+1 \leqslant \alpha \leqslant p+q.$$

The components of the Ricci tensor are obtained by taking the trace of the Riemann curvature tensor over the adapted orthonormal frame fields, $\mathrm{Ric}(F_A) = \sum_{B\neq A} \mathfrak{R}(F_A, F_B, F_B, F_A)$.

Tracing the sectional curvatures over the p vectors of the first fiber and the q vectors of the second fiber yields

$$\begin{aligned}\mathrm{Ric}(F_0) &= \sum_{i=1}^{p} \mathfrak{R}(F_0, F_i, F_i, F_0) + \sum_{\alpha=p+1}^{p+q} \mathfrak{R}(F_0, F_\alpha, F_\alpha, F_0)\\ &= p\left(-\frac{\phi''}{\phi}\right) + q\left(-\frac{\psi''}{\psi}\right).\end{aligned}$$

Factoring out the operators gives the structural radial profile

$$\mathrm{Ric}(F_0) = \left(-p\frac{\phi''}{\phi} - q\frac{\psi''}{\psi}\right) F_0.$$

For a vector field F_i tangent to the first spherical factor $\mathbb{S}^p$, the trace involves the radial direction, the remaining $p-1$ directions within the same fiber, and all q directions from the second fiber

$$\begin{aligned}\mathrm{Ric}(F_i) &= \mathfrak{R}(F_i, F_0, F_0, F_i) + \sum_{j\neq i}^{p} \mathfrak{R}(F_i, F_j, F_j, F_i) + \sum_{\alpha=p+1}^{p+q} \mathfrak{R}(F_i, F_\alpha, F_\alpha, F_i)\\ &= -\frac{\phi''}{\phi} + (p-1)\left(\frac{1-(\phi')^2}{\phi^2}\right) + q\left(-\frac{\phi'\psi'}{\phi\psi}\right).\end{aligned}$$

Combining these terms structurally yields

$$\mathrm{Ric}(F_i) = \left(-\frac{\phi''}{\phi} + (p-1)\frac{1-(\phi')^2}{\phi^2} - q\frac{\phi'\psi'}{\phi\psi}\right) F_i.$$

By symmetry, tracing for a vector field F_α tangent to the second spherical factor $\mathbb{S}^q$ over the radial direction, the p directions of the first fiber, and the remaining $q-1$ directions of its own fiber gives

$$\begin{aligned}\mathrm{Ric}(F_\alpha) &= \mathfrak{R}(F_\alpha, F_0, F_0, F_\alpha) + \sum_{i=1}^{p} \mathfrak{R}(F_\alpha, F_i, F_i, F_\alpha) + \sum_{\beta\neq\alpha}^{p+q} \mathfrak{R}(F_\alpha, F_\beta, F_\beta, F_\alpha)\\ &= -\frac{\psi''}{\psi} + p\left(-\frac{\phi'\psi'}{\phi\psi}\right) + (q-1)\left(\frac{1-(\psi')^2}{\psi^2}\right).\end{aligned}$$

Rearranging the terms to match the standard conventions yields

$$\mathrm{Ric}(F_\alpha) = \left(-\frac{\psi''}{\psi} + (q-1)\frac{1-(\psi')^2}{\psi^2} - p\frac{\phi'\psi'}{\phi\psi}\right) F_\alpha.$$

Consequently, because any arbitrary plane section σ in M can be decomposed into components spanning these basic frame directions, all sectional curvatures on the doubly warped product are bounded as combinations of the five fundamental geometric profiles

$$\left\{-\frac{\phi''}{\phi}, -\frac{\psi''}{\psi}, \frac{1-(\phi')^2}{\phi^2}, \frac{1-(\psi')^2}{\psi^2}, -\frac{\phi'\psi'}{\phi\psi}\right\}.$$

With these general curvature profiles in hand, we can now turn our attention directly to the case of product spheres.

Example 3. *Consider the product manifold* $M = \mathbb{S}^{p+1}_{+} \times \mathbb{S}^{q}$ *equipped with the doubly warped product metric*

$$g = dr^2 + \sin^2 r \, g_{\mathbb{S}^p} + \left(\frac{q-1}{p+m}\right) g_{\mathbb{S}^q},$$

where $r \leqslant \pi/2$ *is the radial distance function on the hemisphere factor* $\mathbb{S}^{p+1}_{+}$. *We choose the potential function* $u(r) = \cos r$ *and set* $\lambda = p + m$. *Under this setting,* M *becomes a compact, oriented* m*-quasi-Einstein manifold with boundary, since* $u = \cos(\pi/2) = 0$ *along* ∂M *and* $u > 0$ *on the interior.*

Let $\{F_0, F_1, \dots, F_{p+q}\}$ *be the adapted local orthonormal frame where* $F_0 = \partial_r$, F_i $(1 \leqslant i \leqslant p)$ *are tangent to* $\mathbb{S}^p$, *and* F_α $(p+1 \leqslant \alpha \leqslant p+q)$ *are tangent to* $\mathbb{S}^q$. *Using the identity* $\nabla_i \nabla_j r = \sin r \cos r \, g_{\mathbb{S}^p}$, *the Hessian of the potential function expands via the chain rule as follows*

$$\begin{aligned}
\nabla_i \nabla_j u &= -\cos r \, dr^2 - \sin r \, \nabla_i \nabla_j r \\
&= -\cos r \, dr^2 - \sin^2 r \cos r \, g_{\mathbb{S}^p} \\
&= -\cos r \left(dr^2 + \sin^2 r \, g_{\mathbb{S}^p}\right) \\
&= -u(dr^2 + \sin^2 r \, g_{\mathbb{S}^p}).
\end{aligned}$$

Evaluating this geometric structure along the distinct frame fields yields

$$\begin{aligned}
&\nabla_{F_0} \nabla_{F_0} u = -u dr^2 = -u \, g(F_0, F_0), \\
&\nabla_{F_i} \nabla_{F_i} u = -u \, \sin^2 r g_{\mathbb{S}^i}(F_i, F_i) = -u \, g(F_i, F_i), \\
&\nabla_{F_\alpha} \nabla_{F_\alpha} u = 0.
\end{aligned}$$

On the other hand, we can evaluate the Ricci tensor by utilizing the general component formulas derived at the beginning of this section. Substituting our specific warping profiles $\phi(r) = \sin r$ *and the constant scale* $\psi(r) = \sqrt{\frac{q-1}{p+m}}$*—which imply* $\phi'' = -\sin(r)$ *and* $\psi' = \psi'' = 0$, *we obtain*

$$\begin{aligned}
\mathrm{Ric}(F_0, F_0) &= -p\frac{(-\sin r)}{\sin r} - q(0) = p = p \, g(F_0, F_0), \\
\mathrm{Ric}(F_i, F_i) &= -\frac{(-\sin r)}{\sin r} + (p-1)\frac{1 - \cos^2 r}{\sin^2 r} - q(0) = 1 + (p-1) = p \, g(F_i, F_i), \\
\mathrm{Ric}(F_\alpha, F_\alpha) &= -0 + (q-1)\frac{1-0}{\psi^2} - p(0) = \frac{q-1}{\psi^2} = (p+m) \, g(F_\alpha, F_\alpha).
\end{aligned}$$

Substituting these frame evaluations into the structural tensor profile $\frac{u}{m}(R_{ij} - \lambda g_{ij})$ *with* $\lambda = p + m$ *along each direction reveals*

$$\begin{aligned}
\frac{u}{m}\big(R_{ij} - \lambda g_{ij}\big)(F_0, F_0) &= \frac{u}{m}\big(p - (p+m)\big)g(F_0, F_0) = -u\, g(F_0, F_0),\\
\frac{u}{m}\big(R_{ij} - \lambda g_{ij}\big)(F_i, F_i) &= \frac{u}{m}\big(p - (p+m)\big)g(F_i, F_i) = -u\, g(F_i, F_i),\\
\frac{u}{m}\big(R_{ij} - \lambda g_{ij}\big)(F_\alpha, F_\alpha) &= \frac{u}{m}\big((p+m) - (p+m)\big)g(F_\alpha, F_\alpha) = 0.
\end{aligned}$$

Comparing the directional evaluations for both fields confirms that $\nabla_i\nabla_j u = \frac{u}{m}(R_{ij} - \lambda g_{ij})$ *holds identically on all frame fields, validating the structural framework.*

Having concluded the analysis of our primary examples, we turn our attention to the foundational properties governing quasi-Einstein manifolds. In what follows, we establish the core concepts and structural identities necessary to complete this chapter.

2.5 Concepts and properties

As a practical tool for our calculations, the D-tensor defined below helps us establish connections among some of our findings.

Definition 2.5.1. *Let* (M^n, g, f) *be a quasi-Einstein manifold , the* D*-tensor is a* $(0,3)$*-tensor field, defined, in local coordinates, by*

$$\begin{aligned}
D_{ijk} = {} & \frac{1}{(n-1)(n-2)}\left(R_{il}\nabla_l f g_{jk} - R_{jl}\nabla_l f g_{ik}\right)\\
& + \frac{1}{n-2}\left(R_{jk}\nabla_i f - R_{ik}\nabla_j f\right)\\
& - \frac{R}{(n-1)(n-2)}\left(\nabla_i f g_{jk} - \nabla_j f g_{ik}\right).
\end{aligned} \tag{2.2}$$

The following results detail the structural identities and symmetry profiles necessary to map the geometric behavior of the D*-tensor.*

Proposition 2.5.1. *The* D*-tensor satisfies two primary geometric properties regarding its components*

(i) *The* D*-tensor is skew-symmetric with respect to its first two indices*

$$D_{ijk} = -D_{jik}; \tag{2.3}$$

(ii) *The* D*-tensor is trace-free in any of two indices, that is*

$$g^{jk}D_{ijk} = g^{ij}D_{ijk} = g^{ik}D_{ijk} = 0. \tag{2.4}$$

Proof. Before starting, take the D_{ijk} defined as

$$\begin{aligned} D_{ijk} = & \frac{1}{(n-1)(n-2)} \left(R_{il} \nabla_l f g_{jk} - R_{jl} \nabla_l f g_{ik} \right) \\ & + \frac{1}{n-2} \left(R_{jk} \nabla_i f - R_{ik} \nabla_j f \right) \\ & - \frac{R}{(n-1)(n-2)} \left(\nabla_i f g_{jk} - \nabla_j f g_{ik} \right). \end{aligned}$$

(i) To verify the skew-symmetry in the first two indices, we compute D_{jik} by swapping the indices i and j in the expression above

$$\begin{aligned} D_{jik} = & \frac{1}{(n-1)(n-2)} \left(R_{jl} \nabla_l f g_{ik} - R_{il} \nabla_l f g_{jk} \right) \\ & + \frac{1}{n-2} \left(R_{ik} \nabla_j f - R_{jk} \nabla_i f \right) \\ & - \frac{R}{(n-1)(n-2)} \left(\nabla_j f g_{ik} - \nabla_i f g_{jk} \right). \end{aligned}$$

Factoring out a negative sign from each of the grouped terms yields

$$\begin{aligned} D_{jik} = - \Bigg[& \frac{1}{(n-1)(n-2)} \left(R_{il} \nabla_l f g_{jk} - R_{jl} \nabla_l f g_{ik} \right) \\ & + \frac{1}{n-2} \left(R_{jk} \nabla_i f - R_{ik} \nabla_j f \right) \\ & - \frac{R}{(n-1)(n-2)} \left(\nabla_i f g_{jk} - \nabla_j f g_{ik} \right) \Bigg]. \end{aligned}$$

The bracketed expression matches D_{ijk} exactly, proving that $D_{ijk} = -D_{jik}$.

(ii) We evaluate the three metric contractions of D_{ijk} systematically

- **Vanishing of $g^{ij}D_{ijk}$ and $g^{ik}D_{ijk}$** From the fundamental skew-symmetry $D_{ijk} = -D_{jik}$ proven in part (i), contracting the symmetric metric tensor with these anti-symmetric slots yields zero immediately by standard index cancellation

$$g^{ij}D_{ijk} = 0 \quad \text{and} \quad g^{ik}D_{ijk} = -g^{ik}D_{jik} = 0.$$

- **Contraction of $g^{jk}D_{ijk}$** Contracting the final two indices with g^{jk}, we apply the geometric traces $g^{jk}g_{jk} = n$, $g^{jk}g_{ik} = \delta_{ji}$, and $g^{jk}R_{jk} = R$

$$\begin{aligned} g^{jk}D_{ijk} = &\frac{1}{(n-1)(n-2)}\left(R_{il}\nabla_l f \cdot n - R_{jl}\nabla_l f\delta_{ji}\right) \\ &+ \frac{1}{n-2}\left(R\nabla_i f - R_{ik}\nabla_k f\right) \\ &- \frac{R}{(n-1)(n-2)}\left(\nabla_i f \cdot n - \nabla_j f\delta_{ji}\right). \end{aligned}$$

Evaluating the Kronecker deltas ($R_{jl}\delta_{ji} = R_{il}$ and $\nabla_j f\delta_{ji} = \nabla_i f$), the expression neatly reduces to

$$\begin{aligned} g^{jk}D_{ijk} =& \frac{1}{(n-1)(n-2)}(n-1)R_{il}\nabla_l f + \frac{1}{n-2}\left(R\nabla_i f - R_{il}\nabla_l f\right) \\ &- \frac{R}{(n-1)(n-2)}(n-1)\nabla_i f \\ =& \frac{1}{n-2}R_{il}\nabla_l f + \frac{1}{n-2}R\nabla_i f - \frac{1}{n-2}R_{il}\nabla_l f - \frac{1}{n-2}R\nabla_i f \\ =& 0. \end{aligned}$$

This completes the proof. □

For a quasi-Einstein manifold, the scalar curvature is constant when $m = 1$. When $m > 1$, we define the scalar function

$$\rho = \frac{1}{m-1}\Big((n-1)\lambda - R\Big),$$

the components of the symmetric $(0, 2)$-tensor P

$$P_{ij} = R_{ij} - \rho g_{ij},$$

and the components of the $(0, 4)$-tensor Q by

$$\begin{aligned} Q_{ijkl} = &R_{ijkl} + \frac{1}{m}\big(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\big) \\ &- \frac{(\lambda+\rho)}{m}\big(g_{ik}g_{jl} - g_{il}g_{jk}\big) \\ = &R_{ijkl} + \frac{1}{m}\left(P_{ik}g_{jl} + P_{jl}g_{ik} - P_{il}g_{jk} - P_{jk}g_{il}\right) + \frac{\rho-\lambda}{m}(g_{ik}g_{jl} - g_{il}g_{jk}). \end{aligned}$$

The definitions provided above function as essential tools throughout our analysis, clarifying our framework and facilitating the proof of several key results.

Chapter 3

Preliminary Results

In this chapter, we establish several key lemmas and preliminary results that serve as the technical foundation required to prove our main results.

3.1 Propositions and key lemmas

This section focuses on preparing the foundational results necessary to prove our main theorems. The proposition below allows us to compute $\nabla_i \rho$, which will be useful in our subsequent proofs.

Proposition 3.1.1. *Let* (M^n, g, f) *be a quasi-Einstein manifold with* $m \neq 1$. *Then we have:*

$$P_{ij}\nabla_j f = -\frac{m}{2}\nabla_i \rho, \textit{ or equivalently } \frac{1}{2}\nabla_i R = \frac{m-1}{m}R_{ij}\nabla_j f + \frac{1}{m}(R-(n-1)\lambda)\nabla_i f; \quad (3.1)$$

$$(\mathrm{div}R)_{kji} = Q_{ijkl}\nabla_l f - \frac{1}{m}(g_{ik}g_{jl} - g_{il}g_{jk})P_{ls}\nabla_s f. \quad (3.2)$$

Proof. From the twice-Contracted second Bianchi Identity, $\frac{1}{2}\nabla_i R = \nabla_j R_{ji}$, and using $\nabla_i |\nabla_j f|^2 = 2\nabla_i\nabla_j f \nabla_j f$, it follows

$$\begin{aligned}\nabla_i R =& 2\nabla_j R_{ji} = 2\left(\frac{1}{m}\Delta f \nabla_i f + \frac{1}{m}\nabla_j\nabla_i f\nabla_j f - \nabla_j\nabla_i\nabla_j f\right)\\ =& \frac{2}{m}\Delta f\nabla_i f + \frac{1}{m}\nabla_i |\nabla_j f|^2 - 2\nabla_j\nabla_i\nabla_j f.\end{aligned}$$

Now note that by the Ricci Identity we have $\nabla_j\nabla_i\nabla_j f = R_{il}\nabla_l f + \nabla_i \Delta f$, and taking the trace of (2.1)

$$\Delta f = n\lambda - R + \frac{1}{m}|\nabla_j f|^2,$$

which gives

$$\nabla_i \Delta f = \frac{1}{m}\nabla_i |\nabla_j f|^2 - \nabla_i R.$$

Combining the facts above, one obtains that

$$\begin{aligned}
\nabla_i R =& \frac{2}{m}\Delta f \nabla_i f + \frac{1}{m}\nabla_i |\nabla_j f|^2 - 2(R_{il}\nabla_l f + \nabla_i \Delta f)\\
=& \frac{2}{m}\Delta f \nabla_i f + \frac{1}{m}\nabla_i |\nabla_j f|^2 - 2R_{il}\nabla_l f - 2\nabla_i \Delta f\\
=& \frac{2}{m}\Delta f \nabla_i f + \frac{1}{m}\nabla_i |\nabla_j f|^2 - 2R_{il}\nabla_l f - 2\left(\frac{1}{m}\nabla_i |\nabla_j f|^2 - \nabla_i R\right)\\
=& \frac{2}{m}\Delta f \nabla_i f - \frac{1}{m}\nabla_i |\nabla_j f|^2 - 2R_{il}\nabla_l f + 2\nabla_i R.\\
=& \frac{2}{m}\Delta f \nabla_i f - \frac{2}{m}\nabla_i \nabla_j f \nabla_j f - 2R_{il}\nabla_l f + 2\nabla_i R,
\end{aligned}$$

where $\nabla_i \nabla_j f \nabla_j f = \lambda \nabla_i f - R_{ij}\nabla_j f + \frac{1}{m}\nabla_i f |\nabla_j f|^2$. So we have

$$\begin{aligned}
\nabla_i R =& 2R_{il}\nabla_l f - \frac{2}{m}\Delta f \nabla_i f + \frac{2}{m}\left(\lambda \nabla_i f - Rij \nabla_j f + \frac{1}{m}\nabla_i f |\nabla_j f|^2\right)\\
=& 2\left(\frac{m-1}{m}R_{ij}\nabla_j f + \frac{1}{m}(\lambda - \Delta f + \frac{1}{m}|\nabla_j f|^2)\nabla_i f\right).
\end{aligned}$$

Since $R = n\lambda - \Delta f + \frac{1}{m}|\nabla_j f|^2$ we finally get

$$\begin{aligned}
\frac{1}{2}\nabla_i R =& \frac{m-1}{m}R_{ij}\nabla_j f + \frac{1}{m}\left(n\lambda - n\lambda + \lambda - \Delta f + \frac{1}{m}|\nabla_j f|^2\right)\nabla_i f\\
=& \frac{m-1}{m}R_{ij}\nabla_j f + \frac{1}{m}\left((1-n)\lambda + n\lambda - \Delta f + \frac{1}{m}|\nabla_j f|^2\right)\nabla_i f\\
=& \frac{m-1}{m}R_{ij}\nabla_j f + \frac{1}{m}(R - (n-1)\lambda)\nabla_i f,
\end{aligned}$$

which proves (3.1).

For (3.2), using that $P_{is}\nabla_s f = R_{is}\nabla_s f - \rho \nabla_i f$, we have

$$\begin{aligned}
&Q_{ijkl}\nabla_l f - \frac{1}{m}(g_{ik}g_{jl} - g_{il}g_{jk})P_{ls}\nabla_s f\\
&= R_{ijkl}\nabla_l f + \frac{1}{m}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il})\nabla_l f\\
&- \frac{\lambda+\rho}{m}(g_{ik}g_{jl} - g_{il}g_{jk})\nabla_l f - \frac{1}{m}(g_{ik}P_{js}\nabla_s f - g_{jk}P_{is}\nabla_s f)\\
&= R_{ijkl}\nabla_l f + \frac{1}{m}(R_{ik}\nabla_j f + g_{ik}R_{jl}\nabla_l f - g_{jk}R_{il}\nabla_l f - R_{jk}\nabla_i f)\\
&- \left(\frac{\lambda}{m} + \frac{\rho}{m}\right)(g_{ik}\nabla_j f - g_{jk}\nabla_i f) - \frac{1}{m}(g_{ik}R_{js}\nabla_s f - g_{jk}R_{is}\nabla_s f)\\
&- \frac{\rho}{m}(g_{jk}\nabla_i f - g_{ik}\nabla_j f)
\end{aligned}$$

$$
\begin{aligned}
&= R_{ijkl}\nabla_l f + \frac{1}{m}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) + \frac{1}{m}(g_{ik}R_{jl}\nabla_l f - g_{jk}R_{il}\nabla_l f)\\
&- \left(\frac{\lambda}{m} + \frac{\rho}{m}\right)(g_{ik}\nabla_j f - g_{jk}\nabla_i f) - \frac{1}{m}(g_{ik}R_{jl}\nabla_l f - g_{jk}R_{il}\nabla_l f)\\
&+ \frac{\rho}{m}(g_{ik}\nabla_j f - g_{jk}\nabla_i f).
\end{aligned}
$$

Thus we have the following identity

$$
\begin{aligned}
&Q_{ijkl}\nabla_l f - \frac{1}{m}(g_{ik}g_{jl} - g_{il}g_{jk})P_{ls}\nabla_s f\\
&\quad = R_{ijkl}\nabla_l f + \frac{1}{m}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) - \frac{\lambda}{m}(g_{ik}\nabla_j f - g_{jk}\nabla_i f).
\end{aligned}
\tag{3.3}
$$

On other hand, by definition $(divR)_{ijk} = \nabla_j R_{ik} - \nabla_k R_{ij}$ where using (2.1) we get

$$
\begin{aligned}
\nabla_j R_{ik} =& \frac{1}{m}(\nabla_j\nabla_i f\nabla_k f + \nabla_i f\nabla_j\nabla_k f) - \nabla_j\nabla_i\nabla_k f,\\
\nabla_k R_{ij} =& \frac{1}{m}(\nabla_k\nabla_i f\nabla_j f + \nabla_i f\nabla_k\nabla_j f) - \nabla_k\nabla_i\nabla_j f.
\end{aligned}
$$

So we have

$$
\begin{aligned}
(divR)_{ijk} &= \nabla_j R_{ik} - \nabla_k R_{ij} = \frac{1}{m}(\nabla_j\nabla_i f\nabla_k f + \nabla_i f\nabla_j\nabla_k f) - \nabla_j\nabla_i\nabla_k f\\
&- \left(\frac{1}{m}(\nabla_k\nabla_i f\nabla_j f + \nabla_i f\nabla_k\nabla_j f) - \nabla_k\nabla_i\nabla_j f\right)\\
&= \frac{1}{m}(\nabla_j\nabla_i f\nabla_k f - \nabla_k\nabla_i f\nabla_j f) + (\nabla_k\nabla_i\nabla_j f - \nabla_j\nabla_i\nabla_k f)\\
&= R_{kjil}\nabla_l f + \frac{1}{m}\left(\left(\lambda g_{ji} - R_{ij} + \frac{1}{m}\nabla_i f\nabla_j f\right)\nabla_k f - \left(\lambda g_{ik} - R_{ik} + \frac{1}{m}\nabla_i f\nabla_k f\right)\nabla_j f\right)\\
&= R_{kjil}\nabla_l f + \frac{1}{m}(R_{ik}\nabla_j f - R_{ij}\nabla_k f) - \frac{\lambda}{m}(g_{ik}\nabla_j f - g_{ij}\nabla_k f),
\end{aligned}
$$

where we have used the Ricci Identity and (2.1). Note that by taking $i \leftrightarrow k$ it follows that

$$
\begin{aligned}
(divR)_{kji} &= R_{ijkl}\nabla_l f + \frac{1}{m}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) - \frac{\lambda}{m}(g_{ik}\nabla_j f - g_{jk}\nabla_i f)\\
&= Q_{ijkl}\nabla_l f - \frac{1}{m}(g_{ik}g_{jl} - g_{il}g_{jk})P_{ls}\nabla_s f.
\end{aligned}
$$

□

Remark 4. Using the result above yields

$$
\begin{aligned}
\frac{1}{2}\nabla_i R &= \frac{m-1}{m}R_{ij}\nabla_j f + \frac{1}{m}(R - (n-1)\lambda)\nabla_i f\\
&= \frac{m-1}{m}R_{ij}\nabla_j f - \frac{1}{m}(m-1)\rho\nabla_i f.
\end{aligned}
$$

Then, organizing the terms we get

$$
R_{ij}\nabla_j f = -\frac{m}{2}\nabla_i\rho + \rho\nabla_i f.
$$

This fact allows us to write the D-tensor as being

$$\begin{aligned} D_{ijk} = &\frac{1}{(n-1)(n-2)}\left[\left(-\frac{m}{2}\nabla_i\rho + \rho\nabla_i f\right)g_{jk} - \left(-\frac{m}{2}\nabla_j\rho + \rho\nabla_j f\right)g_{ik}\right] \\ &+ \frac{1}{n-2}\left(R_{jk}\nabla_i f - R_{ik}\nabla_j f\right) \\ &- \frac{(n-1)\lambda - (m-1)\rho}{(n-1)(n-2)}\left(\nabla_i f g_{jk} - \nabla_j f g_{ik}\right). \end{aligned}$$

Rearranging terms, one obtains that

$$\begin{aligned} D_{ijk} = &\frac{1}{n-2}\left(R_{jk}\nabla_i f - R_{ik}\nabla_j f\right) \\ &- \frac{m\rho - (n-1)\lambda}{(n-1)(n-2)}\left(\nabla_i f g_{jk} - \nabla_j f g_{ik}\right) \\ &- \frac{m}{2(n-1)(n-2)}\left(\nabla_i \rho g_{jk} - \nabla_j \rho g_{ik}\right). \end{aligned}$$

It is important to keep the identity above in mind as a tool for the next proofs.

The lemmas below are completely focused on characterizing the alternative properties and geometric behaviors associated with the D-tensor.

Lemma 3.1.1. *Suppose* (M^n, g, f) *is a quasi-Einstein manifold. The Cotton tensor C, D-tensor and Weyl tensor W satisfy the identity*

$$C_{ijk} = -W_{ijkl}\nabla_l f + \frac{m+n-2}{m}D_{ijk}. \tag{3.4}$$

Proof. By the definition of the Cotton tensor, we have

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}\left(g_{jk}\nabla_i R - g_{ik}\nabla_j R\right).$$

Differentiating (2.1) yields

$$\nabla_i R_{jk} = -\nabla_i\nabla_j\nabla_k f + \frac{1}{m}\left(\nabla_i\nabla_j f\nabla_k f + \nabla_j f\nabla_i\nabla_k f\right),$$

and analogously,

$$\nabla_j R_{ik} = -\nabla_j\nabla_i\nabla_k f + \frac{1}{m}\left(\nabla_j\nabla_i f\nabla_k f + \nabla_i f\nabla_j\nabla_k f\right).$$

Subtracting these two equations and using the symmetry of the Hessian ($\nabla_i\nabla_j f = \nabla_j\nabla_i f$), we obtain

$$\nabla_i R_{jk} - \nabla_j R_{ik} = -\left(\nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f\right) + \frac{1}{m}\left(\nabla_j f\nabla_i\nabla_k f - \nabla_i f\nabla_j\nabla_k f\right).$$

By the Ricci identity, the commutator of the third derivatives of f satisfies

$$\nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f = R_{ijkl}\nabla_l f.$$

Next, isolating the Hessian terms from the fundamental equation gives

$$\begin{aligned}\nabla_i\nabla_k f &= \lambda g_{ik} - R_{ik} + \frac{1}{m}\nabla_i f\nabla_k f,\\ \nabla_j\nabla_k f &= \lambda g_{jk} - R_{jk} + \frac{1}{m}\nabla_j f\nabla_k f.\end{aligned}$$

Substituting these expressions into (1.10), we compute:

$$\begin{aligned}C_{ijk} =& -\left(\nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f\right) + \frac{1}{m}\left(\nabla_j f\nabla_i\nabla_k f - \nabla_i f\nabla_j\nabla_k f\right)\\ &- \frac{1}{2(n-1)}\left(g_{jk}\nabla_i R - g_{ik}\nabla_j R\right)\\ =& -R_{ijkl}\nabla_l f - \frac{1}{2(n-1)}\left(g_{jk}\nabla_i R - g_{ik}\nabla_j R\right)\\ &+ \frac{1}{m}\left[\nabla_j f\left(\lambda g_{ik} - R_{ik} + \frac{1}{m}\nabla_i f\nabla_k f\right) - \nabla_i f\left(\lambda g_{jk} - R_{jk} + \frac{1}{m}\nabla_j f\nabla_k f\right)\right]\\ =& -R_{ijkl}\nabla_l - \frac{1}{2(n-1)}\left(g_{jk}\nabla_i R - g_{ik}\nabla_j R\right) + \frac{\lambda}{m}\left(g_{ik}\nabla_j f - g_{jk}\nabla_i f\right)\\ &+ \frac{1}{m}\left(R_{jk}\nabla_i f - R_{ik}\nabla_j f\right).\end{aligned}$$

Using the standard decomposition of the Riemann tensor, we obtain

$$\begin{aligned}C_{ijk} =& -W_{ijkl}\nabla_l f - \frac{1}{n-2}\big(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\big)\nabla_l f\\ &+ \frac{R}{(n-1)(n-2)}\big(g_{ik}g_{jl} - g_{il}g_{jk}\big)\nabla_l f + \frac{1}{m}\big(R_{jk}\nabla_i f - R_{ik}\nabla_j f\big)\\ &+ \frac{\lambda}{m}\big(g_{ik}\nabla_j f - g_{jk}\nabla_i f\big) - \frac{1}{2(n-1)}\big(\nabla_i R g_{jk} - \nabla_j R g_{ik}\big).\end{aligned}$$

Next, organizing and collecting similar geometric terms, it follows that

$$\begin{aligned}C_{ijk} =& -W_{ijkl}\nabla_l f - \frac{1}{n-2}\big(R_{ik}\nabla_j f - R_{jk}\nabla_i f\big) - \frac{1}{n-2}\big(g_{ik}R_{jl}\nabla_l f - g_{jk}R_{il}\nabla_l f\big)\\ &+ \frac{R}{(n-1)(n-2)}\big(g_{ik}\nabla_j f - g_{jk}\nabla_i f\big) + \frac{\lambda}{m}\big(g_{ik}\nabla_j f - g_{jk}\nabla_i f\big)\\ &+ \frac{1}{m}\big(R_{jk}\nabla_i f - R_{ik}\nabla_j f\big) - \frac{1}{2(n-1)}\big(g_{jk}\nabla_i R - g_{ik}\nabla_j R\big)\\ =& -W_{ijkl}\nabla_l f + \frac{m+n-2}{m(n-2)}\big(R_{jk}\nabla_i f - R_{ik}\nabla_j f\big) - \frac{1}{n-2}\big(g_{ik}R_{jl}\nabla_l f - g_{jk}R_{il}\nabla_l f\big)\\ &- \frac{mR - \lambda(n-1)(n-2)}{m(n-1)(n-2)}\big(g_{jk}\nabla_i f - g_{ik}\nabla_j f\big) - \frac{1}{2(n-1)}\big(g_{jk}\nabla_i R - g_{ik}\nabla_j R\big).\end{aligned}$$

Taking $R=(n-1)\lambda-(m-1)\rho$ and using the contracted identity $R_{il}\nabla^l f=-\frac{m}{2}\nabla_i\rho+\rho\nabla_i f$, the substitution implies that

$$\begin{aligned}
C_{ijk} &= -W_{ijkl}\nabla_l f+\frac{m+n-2}{m(n-2)}\big(R_{jk}\nabla_i f-R_{ik}\nabla_j f\big)\\
&\quad-\frac{\lambda(m+n-2)(n-1)-m(m-1)\rho}{m(n-1)(n-2)}\big(g_{jk}\nabla_i f-g_{ik}\nabla_j f\big)\\
&\quad-\frac{m}{2(n-2)}\big(g_{jk}\nabla_i\rho-g_{ik}\nabla_j\rho\big)-\frac{\rho}{n-2}\big(g_{ik}\nabla_j f-g_{jk}\nabla_i f\big)\\
&\quad-\frac{m-1}{2(n-1)}\big(g_{ik}\nabla_j\rho-g_{jk}\nabla_i\rho\big)\\
&= -W_{ijkl}\nabla_l f+\frac{m+n-2}{m(n-2)}\big(R_{jk}\nabla_i f-R_{ik}\nabla_j f\big)\\
&\quad-\left(\frac{\lambda(m+n-2)(n-1)-m(m-1)\rho}{m(n-1)(n-2)}-\frac{\rho}{n-2}\right)\big(g_{jk}\nabla_i f-g_{ik}\nabla_j f\big)\\
&\quad-\left(\frac{m}{2(n-2)}-\frac{m-1}{2(n-1)}\right)\big(g_{jk}\nabla_i\rho-g_{ik}\nabla_j\rho\big)\\
&= -W_{ijkl}\nabla_l f+\frac{m+n-2}{m(n-2)}\big(R_{jk}\nabla_i f-R_{ik}\nabla_j f\big)\\
&\quad-\frac{m+n-2}{m(n-1)(n-2)}\big(\lambda(n-1)-m\rho\big)\big(g_{jk}\nabla_i f-g_{ik}\nabla_j f\big)\\
&\quad-\frac{m+n-2}{2(n-1)(n-2)}\big(g_{jk}\nabla_i\rho-g_{ik}\nabla_j\rho\big).
\end{aligned}$$

Finally, organizing out the coefficient $(m+n-2)/m$, we conclude that

$$C_{ijk}=-W_{ijkl}\nabla_l f+\frac{m+n-2}{m}D_{ijk},$$

as desired. □

Lemma 3.1.2. *Suppose (M^n,g,f) is a quasi-Einstein manifold. Let $\Sigma^{(n-1)}$ be a level set of f with $\nabla_i f(p)\neq 0$, and let $h_{ab}(a,b=2,\dots,n)$ and $H=(n-1)\sigma$ be its second fundamental form and mean curvature, respectively. Then we have*

$$|D|^2=\frac{2\,|\nabla f|^4}{(n-2)^2}\sum_{a,b=2}^{n}|h_{ab}-\sigma g_{ab}|^2+\frac{m^2}{2(n-1)(n-2)(m-1)^2}\left|\nabla^\Sigma R\right|^2,$$

where $\left|\nabla^\Sigma R\right|^2=|\nabla R|^2-\left\langle\nabla_i R,\nabla_i f\right\rangle^2/\,|\nabla f|^2$.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal frame with $e_1=\nabla_i f/|\nabla f|$ at a point where $\nabla_i f$ is nonzero. First, take the Second fundamental form h_{ab} and the mean curvature H of the level hypersurface Σ as being

$$h_{ab} = \left\langle \nabla_{e_a} \frac{\nabla_i f}{|\nabla f|}, e_b \right\rangle = \frac{1}{|\nabla f|} \langle \nabla_{e_a} \nabla_i f, e_b \rangle = \frac{1}{|\nabla f|} \nabla_{e_a} \nabla_{e_b} f = \frac{1}{|\nabla f|} (\lambda g_{ab} - R_{ab}),$$

$$\begin{aligned} H &= \frac{1}{|\nabla f|} \sum_{a=2}^{n} h_{aa} = \frac{1}{|\nabla f|} \Big(\lambda \sum_{a=2}^{n} g_{aa} - \sum_{a=2}^{n} R_{aa} \Big) = \frac{1}{|\nabla f|} \Big(\lambda(n-1) - \sum_{a=2}^{n} R_{aa} \Big) \\ &= \frac{1}{|\nabla f|} \Big(\lambda(n-1) - (R - R_{11}) \Big), \end{aligned}$$

where

$$R = \sum_{a=1}^{n} R_{aa} = \sum_{a=2}^{n} R_{aa} + R_{11}, \text{ and then } \sum_{a=2}^{n} R_{aa} = R - R_{11}.$$

Now, by using that $R_{il} \nabla_l f = \rho \nabla_i f - \frac{m}{2} \nabla_i \rho$, yields

$$\begin{aligned} R_{11} &= \frac{1}{|\nabla f|^2} R_{ij} \nabla_j f \nabla_i f = \frac{1}{|\nabla f|^2} \Big(\rho \nabla_i f - \frac{m}{2} \nabla_i \rho \Big) \nabla_i f = \rho - \frac{m}{2\,|\nabla f|^2} \big\langle \nabla_i \rho, \nabla_i f \big\rangle, \\ R_{1a} &= \frac{1}{|\nabla f|} R_{aj} \nabla_j f = \frac{\rho}{|\nabla f|} \big\langle \nabla f, e_a \big\rangle - \frac{m}{2\,|\nabla f|} \big\langle \nabla_i \rho, e_a \big\rangle = -\frac{m}{2\,|\nabla f|} \nabla_a \rho. \end{aligned}$$

Moreover, notice that

$$g_{ab} \sigma = \frac{1}{(n-1)} g_{ab} H = \frac{1}{(n-1)\,|\nabla f|} \Big(\lambda(n-1) - (R - R_{11}) \Big) g_{ab}.$$

So we have

$$\begin{aligned} \sum_{a,b=2}^{n} |h_{ab} - \sigma g_{ab}|^2 &= \sum_{a,b=2}^{n} \left| \frac{1}{|\nabla f|} (\lambda g_{ab} - R_{ab}) - \frac{1}{(n-1)\,|\nabla f|} \Big(\lambda(n-1) - (R - R_{11}) \Big) g_{ab} \right|^2 \\ &= \sum_{a,b=2}^{n} \frac{1}{|\nabla f|^2} \left| (\lambda g_{ab} - R_{ab}) - \frac{1}{(n-1)} \Big(\lambda(n-1) - (R - R_{11}) \Big) g_{ab} \right|^2 \\ &= \sum_{a,b=2}^{n} \frac{1}{|\nabla f|^2} \left| \frac{1}{(n-1)} (R - R_{11}) g_{ab} - R_{ab} \right|^2. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{a,b=2}^{n} |h_{ab} - \sigma g_{ab}|^2 \\ &= \sum_{a,b=2}^{n} \frac{1}{|\nabla f|^2} \left(\frac{1}{(n-1)^2} (R - R_{11})^2 g_{ab} g_{ab} - \frac{2}{(n-1)} (R - R_{11}) g_{ab} R_{ab} + |R_{ab}|^2 \right) \\ &= \frac{1}{|\nabla f|^2} \left(\frac{1}{(n-1)^2} (R - R_{11})^2 (n-1) - \frac{2}{(n-1)} (R - R_{11}) \sum_{a=2}^{n} R_{aa} + \sum_{a,b=2}^{n} |R_{ab}|^2 \right) \\ &= \frac{1}{|\nabla f|^2} \left(\frac{1}{(n-1)} (R - R_{11})^2 - \frac{2}{(n-1)} (R - R_{11})^2 + \sum_{a,b=2}^{n} |R_{ab}|^2 \right) \\ &= \frac{1}{|\nabla f|^2} \left(\sum_{a,b=2}^{n} |R_{ab}|^2 - \frac{1}{(n-1)} (R - R_{11})^2 \right). \end{aligned}$$

Now observe that

$$\sum_{a,b=1}^{n}|R_{ab}|^2=\sum_{b=1}^{n}|R_{1b}|^2+\sum_{a=2}^{n}\sum_{b=1}^{n}|R_{ab}|^2=|R_{11}|^2+\sum_{b=2}^{n}|R_{1b}|^2+\sum_{a=2}^{n}\sum_{b=2}^{n}|R_{ab}|^2$$
$$=|R_{11}|^2+\sum_{b=2}^{n}|R_{1b}|^2+\sum_{a=2}^{n}|R_{a1}|^2+\sum_{a=2}^{n}\sum_{b=2}^{n}|R_{ab}|^2,$$

and then

$$\sum_{a,b=2}^{n}|R_{ab}|^2=\sum_{a,b=1}^{n}|R_{ab}|^2-|R_{11}|^2-2\sum_{a=2}^{n}|R_{a1}|^2.$$

Using the fact above it follows

$$\sum_{a,b=2}^{n}|h_{ab}-\sigma g_{ab}|^2$$
$$=\frac{1}{|\nabla f|^2}\left(\sum_{a,b=1}^{n}|R_{ab}|^2-|R_{11}|^2-2\sum_{a=2}^{n}|R_{a1}|^2-\frac{1}{(n-1)}(R-R_{11})^2\right)$$
$$=\sum_{a,b=1}^{n}\frac{1}{|\nabla f|^2}|R_{ab}|^2-\frac{1}{|\nabla f|^2}|R_{11}|^2-\frac{2}{|\nabla f|^2}\sum_{a=2}^{n}|R_{a1}|^2-\frac{1}{(n-1)|\nabla f|^2}(R-R_{11})^2,$$

where

$$-\frac{1}{|\nabla f|^2}|R_{11}|^2=-\frac{1}{|\nabla f|^2}\left(\rho^2-\frac{m\rho}{|\nabla f|^2}\langle\nabla_i\rho,\nabla_i f\rangle+\frac{m^2}{4|\nabla f|^4}\langle\nabla_i\rho,\nabla_i f\rangle^2\right)$$
$$=-\frac{1}{|\nabla f|^2}\rho^2+\frac{m\rho}{|\nabla f|^4}\langle\nabla_i\rho,\nabla_i f\rangle-\frac{m^2}{4|\nabla f|^6}\langle\nabla_i\rho,\nabla_i f\rangle^2,$$

$$-\frac{2}{|\nabla f|^2}\sum_{a=2}^{n}|R_{a1}|^2=-\frac{2}{|\nabla f|^2}\left(-\frac{m}{2|\nabla f|}\sum_{a=2}^{n}\nabla_a\rho\right)^2=-\frac{2}{|\nabla f|^2}\left(\frac{m^2}{4|\nabla f|^2}\sum_{a=2}^{n}|\nabla_a\rho|^2\right)$$
$$=-\frac{2}{|\nabla f|^2}\left(\frac{m^2}{4|\nabla f|^2}\left(\sum_{a=1}^{n}|\nabla_a\rho|^2-|\nabla_1\rho|^2\right)\right)$$
$$=-\frac{2}{|\nabla f|^2}\left(\frac{m^2}{4|\nabla f|^2}\left(|\nabla\rho|^2-\frac{1}{|\nabla f|^2}\langle\nabla_i\rho,\nabla_i f\rangle^2\right)\right)$$
$$=-\frac{m^2}{2|\nabla f|^4}|\nabla\rho|^2+\frac{m^2}{2|\nabla f|^6}\langle\nabla_i\rho,\nabla_i f\rangle^2,$$

$$R-R_{11}=(n-1)\lambda-(m-1)\rho-\left(\rho-\frac{m}{2|\nabla f|^2}\langle\nabla_i\rho,\nabla_i f\rangle\right)$$
$$=(n-1)\lambda-m\rho+\frac{m}{2|\nabla f|^2}\langle\nabla_i\rho,\nabla_i f\rangle,$$

and

$$
\begin{aligned}
(R - R_{11})^2 = (n-1)^2\lambda^2 + m^2\rho^2 + \frac{m^2}{4\,|\nabla_i f|^4}\big\langle \nabla_i\rho, \nabla_i f\big\rangle^2 \\
- 2(n-1)\lambda m\rho + \frac{m(n-1)\lambda}{|\nabla_i f|^2}\big\langle \nabla_i\rho, \nabla_i f\big\rangle - \frac{m^2\rho}{|\nabla_i f|^2}\big\langle \nabla_i\rho, \nabla_i f\big\rangle,
\end{aligned}
$$

hence

$$
\begin{aligned}
-\frac{1}{(n-1)\,|\nabla f|^2}(R - R_{11})^2 &= -\frac{(n-1)\lambda^2}{|\nabla f|^2} - \frac{m^2\rho^2}{(n-1)\,|\nabla f|^2} - \frac{m^2}{4(n-1)\,|\nabla f|^6}\big\langle \nabla_i\rho, \nabla f\big\rangle^2 \\
&\quad + \frac{2(n-1)\lambda m\rho}{(n-1)\,|\nabla f|^2} - \frac{m(n-1)\lambda}{(n-1)\,|\nabla f|^4}\big\langle \nabla_i\rho, \nabla_i f\big\rangle \\
&\quad + \frac{m^2\rho}{(n-1)\,|\nabla f|^4}\big\langle \nabla_i\rho, \nabla_i f\big\rangle \\
&= -\frac{(n-1)\lambda^2}{|\nabla f|^2} - \frac{m^2\rho^2}{(n-1)\,|\nabla f|^2} - \frac{m^2}{4(n-1)\,|\nabla f|^6}\big\langle \nabla_i\rho, \nabla_i f\big\rangle^2 \\
&\quad + \frac{2\lambda m\rho}{|\nabla f|^2} + \frac{m\big(m\rho - (n-1)\lambda\big)}{(n-1)\,|\nabla f|^4}\big\langle \nabla_i\rho, \nabla_i f\big\rangle.
\end{aligned}
$$

Adding them together yields

$$
\begin{aligned}
&\sum_{a,b=2}^{n} |h_{ab} - \sigma g_{ab}|^2 \\
&\quad = \sum_{a,b=1}^{n} \frac{1}{|\nabla f|^2}\,|R_{ab}|^2 - \frac{1}{|\nabla f|^2}\,|R_{11}|^2 - \frac{2}{|\nabla f|^2}\sum_{a=2}^{n}|R_{a1}|^2 - \frac{1}{(n-1)\,|\nabla f|^2}(R - R_{11})^2 \\
&\quad = \sum_{a,b=1}^{n} \frac{1}{|\nabla f|^2}\,|R_{ab}|^2 - \frac{1}{|\nabla f|^2}\rho^2 + \frac{m\rho}{|\nabla f|^4}\big\langle \nabla_i\rho, \nabla_i f\big\rangle - \frac{m^2}{4\,|\nabla f|^6}\big\langle \nabla_i\rho, \nabla_i f\big\rangle^2 \\
&\qquad - \frac{m^2}{2\,|\nabla f|^4}\,|\nabla\rho|^2 + \frac{m^2}{2\,|\nabla f|^6}\big\langle \nabla_i\rho, \nabla_i f\big\rangle^2 - \frac{n-1}{|\nabla f|^2}\lambda^2 \\
&\qquad - \frac{m^2\rho^2}{(n-1)\,|\nabla f|^2} - \frac{m^2}{4(n-1)\,|\nabla f|^6}\big\langle \nabla_i\rho, \nabla_i f\big\rangle^2 + \frac{2\lambda m\rho}{|\nabla f|^2} \\
&\qquad + \frac{m\big(m\rho - (n-1)\lambda\big)}{(n-1)\,|\nabla f|^4}\big\langle \nabla_i\rho, \nabla_i f\big\rangle \\
&\quad = \sum_{a,b=1}^{n} \frac{1}{|\nabla f|^2}\,|R_{ab}|^2 - \frac{m^2}{2\,|\nabla f|^2}\,|\nabla\rho|^2 + \frac{2m^2(n-1) - m^2 - m^2(n-1)}{4(n-1)\,|\nabla f|^2}\big\langle \nabla_i\rho, \nabla_i f\big\rangle^2 \\
&\qquad + \frac{m\rho(n-1) + m(m\rho - n - 1)\lambda}{(n-1)\,|\nabla f|^4}\big\langle \nabla_i\rho, \nabla_i f\big\rangle - \frac{(n-1)\rho^2 - m\rho^2}{(n-1)\,|\nabla f|^2} \\
&\qquad + \frac{2\lambda m\rho}{|\nabla f|^2} - \frac{n-1}{|\nabla f|^2}\lambda^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{|\nabla f|^2}\sum_{a,b=1}^{n}|R_{ab}|^2 - \frac{m^2}{2\,|\nabla f|^4}\,|\nabla\rho|^2 + \frac{m^2(n-2)}{4(n-1)\,|\nabla f|^6}\big\langle \nabla_i\rho, \nabla_i f\big\rangle^2 \\
&\quad + \frac{m\big((m+n-1)\rho-(n-1)\lambda\big)}{(n-1)\,|\nabla f|^4}\big\langle \nabla_i\rho, \nabla f\big\rangle - \frac{m^2+n-1}{(n-1)\,|\nabla f|^2}\rho^2 \\
&\quad + \frac{2m}{|\nabla f|^2}\lambda\rho - \frac{n-1}{|\nabla f|^2}\lambda^2.
\end{aligned}
$$

Now take $D_{ijk} = D(e_i, e_j, e_k)$. Then we have

$$
D_{ijk} = b_1(g_{jk}\nabla_i\rho - g_{ik}\nabla_j\rho) + b_2(\nabla_i f\, R_{jk} - \nabla_j f\, R_{ik}) + b_3(\nabla_i f\, g_{jk} - \nabla_j f\, g_{ik})
$$

where $\nabla_i = \nabla_{e_i}$ and

$$
b_1 = -\frac{m}{2(n-1)(n-2)},\ b_2 = \frac{1}{n-2},\ b_3 = -\frac{(n-1)\lambda - m\rho}{(n-1)(n-2)}.
$$

This gives us that

$$
\begin{aligned}
|D|^2 &= \sum_{i,j,k=1}^{n}|D_{ijk}|^2 \\
&= b_1^2\Big(n\sum_{i,j=1}^{n}|\nabla\rho|^2 - 2\sum_{i=1}^{n}|\nabla\rho|^2 + n\sum_{i,j=1}^{n}|\nabla\rho|^2\Big) \\
&\quad + b_2^2\Big(\sum_{i,j,k=1}^{n}|R_{jk}|^2|\nabla f|^2 - 2\sum_{i,j,k=1}^{n}R_{jk}R_{ik}\nabla_i f\nabla_j f + \sum_{i,j,k=1}^{n}|R_{ik}|^2|\nabla f|^2\Big) \\
&\quad + b_3^2\Big(\sum_{i,j=1}^{n}|\nabla f|^2 - 2\sum_{i=1}^{n}|\nabla f|^2 + \sum_{i,j=1}^{n}|\nabla f|^2\Big) \\
&\quad + 2b_1b_2\Big(\sum_{i,j,k=1}^{n}R_{jk}\delta_{jk}\langle\nabla_i\rho,\nabla_i f\rangle - 2\sum_{i,j=1}^{n}R_{ij}\langle\nabla_i\rho,\nabla_j f\rangle + \sum_{i,j,k=1}^{n}R_{ik}\delta_{ik}\langle\nabla_j\rho,\nabla_j f\rangle\Big) \\
&\quad + 2b_1b_3\Big(n\sum_{i,j=1}^{n}\langle\nabla_i\rho,\nabla_i f\rangle - 2\sum_{i,j=1}^{n}\langle\nabla_i\rho,\nabla_j f\rangle + n\sum_{i,j=1}^{n}\langle\nabla_j\rho,\nabla_j f\rangle\Big) \\
&\quad + 2b_2b_3\Big(\sum_{i,j,k=1}^{n}R_{jk}\delta_{jk}|\nabla f|^2 - 2\sum_{i,j=1}^{n}R_{ij}\langle\nabla_i f,\nabla_j f\rangle + \sum_{i,j,k=1}^{n}R_{ik}\delta_{ik}|\nabla f|^2\Big),
\end{aligned}
$$

so we have

$$
\begin{aligned}
|D|^2 &= b_1^2\big(2(n-1)\,|\nabla\rho|^2\,\big) + b_2^2\big(2\sum_{i,k=1}^{n}|R_{ik}|^2\,|\nabla f|^2 - 2\sum_{i,k=1}^{n}|R_{ik}\nabla_i f|^2\,\big) \\
&\quad + b_3^2\big(2(n-1)\,|\nabla f|^2\,\big) + 2b_1b_2\big(2R\big\langle\nabla_i\rho,\nabla_i f\big\rangle - 2\sum_{i,j=1}^{n}R_{ij}\big\langle\nabla_i\rho,\nabla_j f\big\rangle\big) \\
&\quad + 2b_1b_3\big(2(n-1)\big\langle\nabla_i\rho,\nabla_i f\big\rangle\big) + 2b_2b_3\big(2R\,|\nabla f|^2 - \sum_{i,j}2R_{ij}\big\langle\nabla_i f,\nabla_j f\big\rangle\big).
\end{aligned}
$$

Now note that by $R_{il}\nabla_l f = -\frac{m}{2}\nabla_i\rho + \rho\nabla_i f$, it follows

$$\sum_{i,j=1}^{n} R_{ij}\big\langle \nabla_j f, \nabla_i\rho\big\rangle = \sum_{ij}^{n}\big\langle R_{ij}\nabla_j f, \nabla_i\rho\big\rangle = \sum_{i,j=1}^{n}\big\langle -\frac{m}{2}\nabla_i\rho + \rho\nabla_i f, \nabla_i\rho\big\rangle$$
$$= -\frac{m}{2}\left|\nabla\rho\right|^2 + \rho\big\langle\nabla_i f, \nabla_i\rho\big\rangle,$$

$$\sum_{i,k=1}^{n}\left|R_{ik}\nabla_k f\right|^2 = \sum_{i=1}^{n}\big\langle -\frac{m}{2}\nabla_i\rho + \rho\nabla_i f, -\frac{m}{2}\nabla_i\rho + \rho\nabla_i f\big\rangle$$
$$= \frac{m^2}{4}\left|\nabla\rho\right|^2 - m\rho\big\langle\nabla_i\rho, \nabla_i f\big\rangle + \rho^2\left|\nabla f\right|^2,$$

$$\sum_{i,j=1}^{n} R_{ij}\big\langle\nabla_i f, \nabla_j f\big\rangle = -\frac{m}{2}\big\langle\nabla_i\rho, \nabla_i f\big\rangle + \rho\left|\nabla f\right|^2,$$

and then

$$\begin{aligned}
|D|^2 = {} & \frac{m^2}{4(n-1)^2(n-2)^2}\big(2(n-1)\left|\nabla\rho\right|^2\big) \\
& + \frac{1}{(n-2)^2}\Big(2\sum_{i,k=1}^{n}\left|R_{ik}\right|^2\left|\nabla f\right|^2 - 2\big(\frac{m^2}{4}\left|\nabla\rho\right|^2 - m\rho\big\langle\nabla_i\rho, \nabla_i f\big\rangle + \rho^2\left|\nabla f\right|^2\big)\Big) \\
& + \frac{\big((n-1)\lambda - m\rho\big)^2}{(n-1)^2(n-2)^2}\big(2(n-1)\left|\nabla f\right|^2\big) \\
& - \frac{2m}{2(n-1)(n-2)^2}\Big(2R\big\langle\nabla_i\rho, \nabla_i f\big\rangle - 2\big(-\frac{m}{2}\left|\nabla\rho\right|^2 + \rho\big\langle\nabla_i f, \nabla_i\rho\big\rangle\big)\Big) \\
& + \frac{2m((n-1)\lambda - m\rho)}{2(n-1)^2(n-2)^2}\big(2(n-1)\big\langle\nabla_i\rho, \nabla_i f\big\rangle\big) \\
& - \frac{2\big((n-1)\lambda - m\rho)\big)}{(n-1)(n-2)}\Big(2R\left|\nabla f\right|^2 - 2\big(-\frac{m}{2}\big\langle\nabla_i\rho, \nabla_i f\big\rangle + \rho\left|\nabla f\right|^2\big)\Big).
\end{aligned}$$

Writing the expression over a common denominator

$$\begin{aligned}
|D|^2 = {} & \frac{m^2}{2(n-1)(n-2)^2}\left|\nabla\rho\right|^2 \\
& + \frac{2(n-1)}{2(n-1)(n-2)^2}\Big(2\sum_{i,k=1}^{n}\left|R_{ik}\right|^2\left|\nabla f\right|^2 - 2\big(\frac{m^2}{4}\left|\nabla\rho\right|^2 - m\rho\big\langle\nabla_i\rho, \nabla_i f\big\rangle + \rho^2\left|\nabla f\right|^2\big)\Big) \\
& + \frac{2\big((n-1)\lambda - m\rho\big)^2}{2(n-1)(n-2)^2}\big(2\left|\nabla f\right|^2\big) \\
& - \frac{2m}{2(n-1)(n-2)^2}\Big(2R\big\langle\nabla_i\rho, \nabla_i f\big\rangle - 2\big(-\frac{m}{2}\left|\nabla\rho\right|^2 + \rho\big\langle\nabla_i f, \nabla_i\rho\big\rangle\big)\Big) \\
& + \frac{2m((n-1)\lambda - m\rho)}{2(n-1)(n-2)^2}\big(2\big\langle\nabla_i\rho, \nabla_i f\big\rangle\big) \\
& - \frac{4\big((n-1)\lambda - m\rho)\big)}{2(n-1)(n-2)}\Big(2R\left|\nabla f\right|^2 - 2\big(-\frac{m}{2}\big\langle\nabla_i\rho, \nabla_i f\big\rangle + \rho\left|{}^{\nabla} f\right|^2\big)\Big),
\end{aligned}$$

so we have

$$
\begin{aligned}
|D|^2 &= \frac{m^2-(n-1)m^2-2m^2}{2(n-1)(n-2)^2}|\nabla\rho|^2 + \frac{4(n-1)}{2(n-1)(n-2)^2}|\nabla f|^2\sum_{i,j=1}^{n}|R_{ij}|^2 \\
&\quad + \frac{4m\big(\rho(n-1)-R+\rho-\big((n-1)\lambda-m\rho\big)+\big((n-1)\lambda-m\rho\big)\big)}{2(n-1)(n-1)^2}\big\langle\nabla_i\rho,\nabla_i f\big\rangle \\
&\quad - \frac{4\big(\rho^2(n-1)-\big((n-1)\lambda-m\rho\big)^2-\rho\big((n-1)\lambda-m\rho\big)+\big((n-1)\lambda-m\rho\big)2R\big)}{2(n-1)(n-2)^2}|\nabla f|^2 \\
&= -\frac{m^2 n}{2(n-1)(n-2)^2}|\nabla\rho|^2 + \frac{4(n-1)}{2(n-1)(n-2)^2}|\nabla f|^2\sum_{i,j=1}^{n}|R_{ij}|^2 \\
&\quad + \frac{4m\big(\rho n-((n-1)\lambda-(m-1)\rho)\big)}{2(n-1)(n-1)^2}\big\langle\nabla_i\rho,\nabla_i f\big\rangle \\
&\quad - \frac{4\big(\rho^2(n-1)+\big((n-1)\lambda-m\rho\big)\big(2\big((n-1)\lambda-m\rho\big)-\big((n-1)\lambda-m\rho\big)\big)\big)}{2(n-1)(n-2)^2}|\nabla f|^2 \\
&= \frac{1}{2(n-1)(n-2)^2}\Big(4(n-1)|\nabla f|^2\sum_{i,j=1}^{2}|R_{ij}|^2 - m^2 n|\nabla\rho|^2\Big) \\
&\quad + \frac{4m\big((m+n-1)\rho-(n-1)\lambda\big)}{2(n-1)(n-1)^2}\big\langle\nabla_i\rho,\nabla_i f\big\rangle \\
&\quad - \frac{4\big(((n-1)\lambda-m\rho)^2+\rho^2(n-1)\big)}{2(n-1)(n-2)^2}|\nabla f|^2 .
\end{aligned}
$$

Now note that

$$
\begin{aligned}
\left|\nabla_i^{\Sigma}R\right|^2 &= |\nabla_i R|^2 - \frac{1}{|\nabla f|^2}\big\langle\nabla_i R,\nabla_i f\big\rangle^2 \\
&= (m-1)^2|\nabla_i\rho|^2 - \frac{(m-1)^2}{|\nabla_i f|^2}\big\langle\nabla_i\rho,\nabla_i f\big\rangle^2 \\
&= (m-1)^2\left|\nabla_i^{\Sigma}\rho\right|^2,
\end{aligned}
$$

which implies that

$$
\left|\nabla_i^{\Sigma}\rho\right|^2 = \frac{1}{(m-1)^2}\left|\nabla_i^{\Sigma}R\right|^2,
$$

and then

$$
\frac{m^2}{2(n-1)(n-2)(m-1)^2}|\nabla^{\Sigma}R|^2 = \frac{m^2}{2(n-1)(n-2)}|\nabla^{\Sigma}\rho|^2.
$$

Now by substituting the terms in $\sum_{a,b=2}^{n}|h_{ab}-\sigma g_{ab}|^2$, it follows that

$$
\begin{aligned}
&\frac{2|\nabla f|^4}{(n-2)^2}\sum_{a,b=2}^{n}|h_{ab}-\sigma g_{ab}|^2 + \frac{m^2}{2(n-1)(n-2)(m-1)^2}|\nabla^{\Sigma}R|^2 \\
&= \frac{2|\nabla f|^4}{(n-2)^2}\sum_{a,b=2}^{n}|h_{ab}-\sigma g_{ab}|^2 + \frac{m^2}{2(n-1)(n-2)}|\nabla^{\Sigma}\rho|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2|\nabla f|^4}{(n-2)^2}\Bigg(\frac{1}{|\nabla f|^2}\sum_{a,b=1}^{n}|R_{ab}|^2 - \frac{m^2}{2|\nabla f|^4}|\nabla\rho|^2 + \frac{m^2(n-2)}{4(n-1)|\nabla f|^6}\big\langle\nabla_i\rho,\nabla_i f\big\rangle^2 \\
&\quad + \frac{m\big((m+n-1)\rho-(n-1)\lambda\big)}{(n-1)|\nabla f|^4}\big\langle\nabla_i\rho,\nabla_i f\big\rangle - \frac{m^2+n-1}{(n-1)|\nabla f|^2}\rho^2 + \frac{2m}{|\nabla f|^2}\lambda\rho - \frac{n-1}{|\nabla f|^2}\lambda^2\Bigg) \\
&\quad + \frac{m^2}{2(n-1)(n-2)}\Big(|\nabla\rho|^2 - \frac{1}{|\nabla f|^2}\big\langle\nabla_i\rho,\nabla_i f\big\rangle^2\Big) \\
&= \frac{2}{(n-2)^2}|\nabla f|^2\sum_{i,j=1}^{n}|R_{ij}|^2 - \frac{2m^2}{2(n-2)^2}|\nabla\rho|^2 + \frac{m^2(n-2)}{2(n-1)(n-2)^2|\nabla f|^2}\langle\nabla_i\rho,\nabla_i f\rangle^2 \\
&\quad + \frac{2m\big((m+n-1)\rho-(n-1)\lambda\big)}{(n-1)(n-2)^2}\langle\nabla_i\rho,\nabla_i f\rangle - \frac{2(m+n-1)\rho^2}{(n-1)(n-2)^2}|\nabla f|^2 \\
&\quad + \frac{4m\lambda\rho}{(n-2)^2}|\nabla f|^2 - \frac{2(n-1)\lambda^2}{(n-2)^2}|\nabla f|^2 \\
&\quad + \frac{m^2}{2(n-1)(n-2)}|\nabla\rho|^2 - \frac{m^2}{2(n-1)(n-2)|\nabla f|^2}\langle\nabla_i\rho,\nabla_i f\rangle^2.
\end{aligned}
$$

Writing the terms over the same denominator yields

$$
\begin{aligned}
&\frac{2|\nabla f|^4}{(n-2)^2}\sum_{a,b=2}^{n}|h_{ab}-\sigma g_{ab}|^2 + \frac{m^2}{2(n-1)(n-2)(m-1)^2}|\nabla^{\Sigma}R|^2 \\
&= \frac{4(n-1)}{2(n-1)(n-2)^2}|\nabla f|^2\sum_{i,j=1}^{n}|R_{ij}|^2 - \frac{2m^2(n-1)}{2(n-1)(n-2)^2}|\nabla\rho|^2 \\
&\quad + \frac{m^2(n-2)}{2(n-1)(n-2)^2|\nabla f|^2}\langle\nabla_i\rho,\nabla_i f\rangle^2 + \frac{4m\big((m+n-1)\rho-(n-1)\lambda\big)}{2(n-1)(n-2)^2}\langle\nabla_i\rho,\nabla_i f\rangle \\
&\quad - \frac{4(m+n-1)\rho^2}{2(n-1)(n-2)^2}|\nabla f|^2 + \frac{8m(n-1)\lambda\rho}{2(n-1)(n-2)^2}|\nabla f|^2 - \frac{4(n-1)^2\lambda^2}{2(n-1)(n-2)^2}|\nabla f|^2 \\
&\quad + \frac{m^2(n-2)}{2(n-1)(n-2)^2}|\nabla\rho|^2 - \frac{m^2(n-2)}{2(n-1)(n-2)^2|\nabla f|^2}\langle\nabla_i\rho,\nabla_i f\rangle^2 \\
&= \frac{4(n-1)}{2(n-1)(n-2)^2}|\nabla f|^2\sum_{i,j=1}^{n}|R_{ij}|^2 + \frac{m^2(n-2)-2(n-1)m^2}{2(n-1)(n-2)^2}|\nabla\rho|^2 \\
&\quad + \frac{4m\big((m+n-1)\rho-(n-1)\lambda\big)}{2(n-1)(n-2)^2}\langle\nabla_i\rho,\nabla_i f\rangle \\
&\quad - \frac{4((n-1)\rho^2+m^2\rho^2-2m\rho(n-1)\lambda+(n-1)^2\lambda^2)}{2(n-1)(n-2)^2}|\nabla f|^2 \\
&= \frac{1}{2(n-1)(n-2)^2}\Big(4(n-1)|\nabla f|^2\sum_{i,j=1}^{n}|R_{ij}|^2 - m^2 n|\nabla\rho|^2\Big) \\
&\quad + \frac{4m\big((m+n-1)\rho-(n-1)\lambda\big)}{2(n-1)(n-2)^2}\langle\nabla_i\rho,\nabla_i f\rangle \\
&\quad - \frac{4\big((m\rho-(n-1)\lambda)^2+(n-1)\rho^2\big)}{2(n-1)(n-2)^2}|\nabla f|^2.
\end{aligned}
$$

Therefore, we conclude that

$$|D|^2 = \frac{2\,|\nabla f|^4}{(n-2)^2}\sum_{a,b=2}^{n}|h_{ab}-\sigma g_{ab}|^2 + \frac{m^2}{2(n-1)(n-2)(m-1)^2}\left|\nabla^{\Sigma}R\right|^2.$$

□

By using **Lemmas** (3.1.1) and (3.1.2) , we are now able to establish the subsequent proposition

Proposition 3.1.2. *Suppose* (M^n, g, f), $n \geqslant 3$, *and a quasi-Einstein manifold with* $m \neq 1$ *and* $D = 0$*. Let c be a regular value of f and* $\Sigma = \{x \in M; f(x) = c\}$ *be the level hypersurface of f. Then:*

(i) *The scalar curvature and* $|\nabla f|^2$ *are constant on* Σ*;*

(ii) *The second fundamental form* h_{ab} *of* Σ *is of the form* $h_{ab} = \frac{H}{(n-1)}g_{ab}$*, and the mean curvature* H *is constant on* Σ*;*

(iii) $R_{abcl}\nabla_l f = 0$*, where* $a, b, c \geqslant 2$ *on* Σ*;*

(iv) *On* Σ*, the Ricci tensor has either a unique eigenvalue or two distinct eigenvalues with multiplicity* 1 *and* $n-1$*; moreover, the eigenvalue with multiplicity* 1 *is in the direction of* ∇f*.*

Proof. To prove (1), take an orthonormal frame $\{e_i\}_{i=1}^n$, with $e_1 = \nabla f / |\nabla f|$, and consider a point where ∇f is nonzero. Using $D_{ijk} = 0$ and **Lemma 3.1.2**, we obtain

$$|\nabla^{\Sigma}R|^2 = 0.$$

Hence,

$$\nabla_a R = 0, \qquad (a = 2, \dots, n),$$

which proves that R is constant on Σ.

Now taking $\nabla_i f = e_1\,|\nabla_i f|$ it follows

$$\nabla_a\left(|\nabla_i f|^2\right) = 2\left\langle \nabla_a\nabla_i f, \nabla_i f\right\rangle = 2\,|\nabla_i f|\left\langle \nabla_a\nabla_i f, e_1\right\rangle$$
$$2\,|\nabla_i f|\,\nabla_a\nabla_1 f.$$

From the fundamental equation

$$\nabla_a\nabla_1 f = \lambda g_{a1} - R_{1a} + \frac{1}{m}\left\langle \nabla_a f, \nabla_1 f\right\rangle$$
$$= -R_{1a},$$

where, because of $R = (n-1)\lambda - (m-1)\rho$ and $\nabla_a R = 0$, we get

$$R_{1a} = -\frac{m}{2\,|\nabla_i f|}\nabla_a \rho = -\frac{m}{2(m-1)\,|\nabla_i f|}\nabla_a R = 0.$$

Hence

$$\nabla_a\left(|\nabla_i f|^2\right) = 2\,|\nabla_i f|\,\nabla_a \nabla_1 f = -2\,|\nabla_i f|\,R_{1a} = 0.$$

For (2), from $D_{ijk} = 0$, and **Lemma 3.1.2** we get

$$|h_{ab} - \sigma g_{ab}| = 0,$$

then we have

$$h_{ab} = \sigma g_{ab} = \frac{1}{n-1} H g_{ab},$$

as claimed.

Using the Codazzi equation, we obtain

$$\nabla_c h_{ab} - \nabla_b h_{ac} = R_{abc1}.$$

Contracting with g^{ab}, it follows that

$$\nabla_c h_{aa} - g^{ab}\nabla_b h_{ac} = R_{c1} = 0.$$

Interchanging the indices a and b, we get

$$\begin{aligned}\nabla_c h_{aa} &= g^{ab}\nabla_a h_{bc}\\ &= g^{ab}\nabla_a(\sigma g_{bc})\\ &= \delta_c^a \nabla_a \sigma\\ &= \nabla_c \sigma.\end{aligned}$$

Hence,

$$\nabla_c h_{aa} - \nabla_c \sigma = \nabla_c H - \frac{1}{n-1}\nabla_c H = \frac{n-2}{n-1}\nabla_c H = 0.$$

Therefore

$$\nabla_c H = 0.$$

This completes the proof of (2).

For the third item, observe that, by the previous results,

$$\nabla_c h_{ab} = \frac{1}{n-1}\nabla_c H\, g_{ab} = 0.$$

Hence,

$$\nabla_c h_{ab} = 0.$$

Similarly,

$$\nabla_b h_{ac} = 0.$$

Consequently, the Codazzi equation implies that

$$\frac{1}{|\nabla f|} R_{abci} \nabla_i f = \nabla_c h_{ab} - \nabla_b h_{ac} = 0,$$

which means that, $R_{abcl} \nabla_l f = 0$.

Finally, by the fundamental equation, we have

$$\nabla_a \nabla_b f = \lambda g_{ab} - R_{ab} + \frac{1}{m} \langle \nabla_a f, \nabla_b f \rangle = \lambda g_{ab} - R_{ab},$$

since $\nabla_a f = 0$ on Σ.

Moreover,

$$h_{ab} = \frac{1}{|\nabla f|} \big(\lambda g_{ab} - R_{ab} \big) = \frac{\nabla_a \nabla_b f}{|\nabla f|},$$

and

$$h_{ab} = \frac{1}{n-1} H g_{ab}.$$

As a consequence,

$$\frac{1}{n-1} H |\nabla f| \, g_{ab} = \nabla_a \nabla_b f = \lambda g_{ab} - R_{ab},$$

which implies that

$$R_{ab} = \left(\lambda - \frac{1}{n-1} H |\nabla f| \right) g_{ab} = \mu g_{ab},$$

where, on Σ,

$$\mu = \lambda - \frac{1}{n-1} H |\nabla f|$$

is constant by (1) and (2).

Considering now the direction of ∇f, it follows that

$$\begin{aligned} R &= R_{11} + \sum_{a=2}^{n} R_{aa} \\ &= R_{11} + (n-1)\mu \\ &= R_{11} + (n-1)\lambda - H|\nabla f|. \end{aligned}$$

Hence,

$$R_{11} = R - (n-1)\lambda + H|\nabla f|.$$

This completes the proof.

$\square$

Chapter 4

Main theorems

This chapter is dedicated to proving two key rigidity results for compact quasi-Einstein manifolds under the assumption of a vanishing Bach tensor. Initially, we establish **Theorem 4.1.1**, showing that a vanishing Bach tensor forces the Weyl tensor to be harmonic and orthogonal to the gradient of the potential function. Second, we specialize to the four-dimensional setting in **Theorem 4.2.1**, where this geometric constraint yields an even stronger rigidity phenomenon, ensuring that the manifold is locally conformally flat.

4.1 General case $(n \geqslant 4)$

We first examine the general case, the proof of which requires several of the main results established in the previous chapter

Theorem 4.1.1. *Suppose* $(M^n, g, f)(n \geqslant 4)$ *is a compact quasi-Einstein manifold with* $m \neq 0, 1,$ *or* $2 - n$*. If the Bach tensor B vanishes everywhere, then* M *has a harmonic Weyl tensor and* $W_{ijkl}\nabla_l f = 0$*.*

Proof. Before starting, fix a point $p \in M$ and let $\{e_k\}_{k=1}^n$ be an orthonormal frame with $\nabla e_i(p) = 0$. Using definition of the Bach tensor we get

$$(n-2)B_{ij} = (\nabla_k C)_{kij} + R_{kl}W_{ikjl},$$

where the Cotton tensor components are given by

$$C_{kij} = -W_{kijl}\nabla_l f + \frac{m+n-2}{m}D_{kij}.$$

Differentiating this relation, we obtain

$$\begin{aligned}(n-2)B_{ij} &= \nabla_k\left(-W_{kijl}\nabla_l f + D_{kij}\right) + R_{kl}W_{ikjl}\\ &= \nabla_k\left(W_{ikjl}\nabla_l f + D_{kij}\right) + R_{kl}W_{ikjl}\\ &= (\nabla_k W)_{ikjl}\nabla_l f + \frac{m+n-2}{m}(\nabla_k D)_{kij} + W_{ikjl}\big(R_{kl} + \nabla_k\nabla_l f\big).\end{aligned}$$

Recall that the fundamental equation of a quasi-Einstein manifold states that

$$R_{kl} + \nabla_k\nabla_l f = \lambda g_{kl} + \frac{1}{m}\nabla_k f\nabla_l f.$$

Substituting this expression back into our equation for the Bach tensor gives

$$(n-2)B_{ij} = (\nabla_k W)_{ikjl}\nabla_l f + \frac{m+n-2}{m}(\nabla_k D)_{kij} + W_{ikjl}\big(\lambda g_{kl} + \frac{1}{m}\nabla_k f\nabla_l f\big).$$

Next, we exploit the algebraic symmetries of the Weyl tensor. Since

$$W_{ikjl} = W_{jlik} = -W_{ljik},$$

taking the divergence with respect to the index a implies that

$$(\nabla_k W)_{ikjl} = -(\nabla_k W)_{ljik} = -(\mathrm{div}W)_{lji}.$$

by the **Proposition 1.2.9**, we get that W_{ikjl} is trace-free and then

$$\lambda g_{kl}W_{ikjl} = 0.$$

Furthermore, using the **Proposition 1.2.10**

$$-(\mathrm{div}W)_{lji}\nabla_l f = \frac{n-3}{n-2}C_{lji}\nabla_l f.$$

Finally, applying these geometric simplifications to the Bach tensor expression, we arrive at

$$\begin{aligned}(n-2)B_{ij} &= -(\mathrm{div}W)_{lji}\nabla_l f + \frac{m+n-2}{m}(\nabla_k D)_{kij}\\ &\quad + \lambda g_{kl}W_{ikjl} + \frac{1}{m}W_{ikjl}\nabla_k f\nabla_l f\\ &= -(\mathrm{div}W)_{lji}\nabla_l f + \frac{m+n-2}{m}(\nabla_k D)_{kij} + \frac{1}{m}W_{ikjl}\nabla_k f\nabla_l f\\ &= \frac{n-3}{n-2}C_{lji}\nabla_l f + \frac{m+n-2}{m}(\nabla_k D)_{kij} + \frac{1}{m}W_{ikjl}\nabla_k\nabla_l f.\end{aligned}$$

By the algebraic skew-symmetries of the Cotton and Weyl tensors, we observe that

$$C_{lji}\nabla_l f\nabla_j f = -C_{jli}\nabla_j f\nabla_l f = -C_{lji}\nabla_l f\nabla_j f,$$

which immediately implies this contraction vanishes. By a similar argument using the pairwise skew-symmetries of the Weyl tensor, it follows that

$$W_{ikjl}\nabla_i f\nabla_k f\nabla_j f\nabla_l f = 0.$$

Hence, contracting both sides of our main identity with $\nabla_i f$ and $\nabla_j f$, we obtain

$$(n-2)B_{ij}\nabla_i f\nabla_j f = \frac{m+n-2}{m}(\nabla_k D)_{kij}\nabla_i f\nabla_j f.$$

Now, on integrating over M, we infer

$$\frac{m(n-2)}{m+n-2}\int_M B_{ij}\nabla_i f\nabla_j f\, dV_g = \int_M (\nabla_k D)_{kij}\nabla_i f\nabla_j f\, dV_g. \tag{4.1}$$

Applying the product rule to the divergence of the vector field $D_{kij}\nabla_i f\nabla_j f$, we obtain

$$\nabla_k\big(D_{kij}\nabla_i f\nabla_j f\big) = \big(\nabla_k D\big)_{kij}\nabla_i f\nabla_j f + D_{kij}\nabla_k\nabla_i f\nabla_j f + D_{kij}\nabla_i f\nabla_k\nabla_j f,$$

Using the skew-symmetry of the D-tensor in its first two indices ($D_{kij} = -D_{ikj}$), we observe that

$$D_{kij}\nabla_k\nabla_i f\nabla_j f = D_{kij}\nabla_i\nabla_k f\nabla_j f = -D_{ikj}\nabla_i\nabla_k f\nabla_j f.$$

Relabeling the dummy indices $i \leftrightarrow k$ on the right-hand side, it follows that

$$D_{kij}\nabla_k\nabla_i f\nabla_j f = -D_{kij}\nabla_k\nabla_i f\nabla_j f, \quad \text{which implies} \quad D_{kij}\nabla_k\nabla_i f\nabla_j f = 0.$$

Integrating the divergence expression over M, the second term on the right-hand side vanishes identically, leading to

$$\int_M \nabla_k\big(D_{kij}\nabla_i f\nabla_j f\big)\, dV_g = \int_M (\nabla_k D)_{kij}\nabla_i f\nabla_j f\, dV_g + \int_M D_{kij}\nabla_i f\nabla_k\nabla_j f\, dV_g.$$

Assuming M is a closed manifold, the Divergence Theorem implies that

$$\int_M \nabla_k\big(D_{kij}\nabla_i f\nabla_j f\big)\, dV_g = 0.$$

Consequently, we establish the following integration-by-parts relation

$$\int_M (\nabla_k D)_{kij}\nabla_i f\nabla_j f\, dV_g = -\int_M D_{kij}\nabla_i f\nabla_k\nabla_j f\, dV_g.$$

Combining these facts, we finally arrive at

$$\frac{m(n-2)}{m+n-2}\int_M B_{ij}\nabla_i f\nabla_j f\, dV_g = -\int_M D_{kij}\nabla_i f\nabla_k\nabla_j f\, dV_g.$$

Using the fundamental equation to substitute the Hessian term $\nabla_k \nabla_j f$, and recalling that D_{kij} is completely trace-free ($g_{kj} D_{kij} = 0$) and satisfies $D_{kij} \nabla_k f \nabla_j f = 0$, we obtain

$$\begin{aligned} -D_{kij} \nabla_i f \nabla_k \nabla_j f &= D_{kij} \nabla_i f \big(R_{kj} - \lambda g_{kj} - \frac{1}{m} \nabla_k f \nabla_j f \big) \\ &= D_{kij} R_{kj} \nabla_i f - \lambda g_{kj} D_{kij} - \frac{1}{m} D_{kij} \nabla_k f \nabla_j f \\ &= D_{kij} R_{kj} \nabla_i f. \end{aligned}$$

Next, consider the following combination

$$\frac{1}{2} D_{kij} \big(R_{kj} \nabla_i f - R_{ij} \nabla_k f \big) = \frac{1}{2} D_{kij} R_{kj} \nabla_i f - \frac{1}{2} D_{kij} R_{ij} \nabla_k f.$$

By relabeling the dummy indices $k \leftrightarrow i$ and invoking the skew-symmetry of D_{kij} in its first two indices ($D_{kij} = -D_{ikj}$), the second term on the right-hand side transforms as follows

$$\begin{aligned} \frac{1}{2} D_{kij} \big(R_{kj} \nabla_i f - R_{ij} \nabla_k f \big) &= \frac{1}{2} D_{kij} R_{kj} \nabla_i f - \frac{1}{2} D_{ikj} R_{kj} \nabla_i f \\ &= \frac{1}{2} D_{kij} R_{kj} \nabla_i f + \frac{1}{2} D_{kij} R_{kj} \nabla_i f \\ &= D_{kij} R_{kj} \nabla_i f. \end{aligned}$$

Furthermore, by the standard definition of the quasi-Einstein invariant tensor D_{aij}, we have

$$\begin{aligned} D_{kij} = &-\frac{m}{2(n-1)(n-2)} \big(\nabla_k \rho g_{ij} - \nabla_i \rho g_{kj} \big) + \frac{1}{n-2} \big(R_{ij} \nabla_k f - R_{kj} \nabla_i f \big) \\ &- \frac{(n-1)\lambda - m\rho}{(n-1)(n-2)} \big(g_{ij} \nabla_k f - g_{kj} \nabla_i f \big). \end{aligned}$$

Contraction of this definition directly with D_{aij} yields a simplification where the terms proportional to the metric g vanish due to the trace-free property of D. This leads directly to the relation

$$-(n-2)|D|^2 = D_{kij} \big(R_{aj} \nabla_i f - R_{ij} \nabla_k f \big).$$

Consequently, we can rewrite the initial pointwise contraction entirely in terms of the norm of D

$$\begin{aligned} -D_{kij} \nabla_i f \nabla_k \nabla_j f &= D_{kij} R_{kj} \nabla_i f \\ &= \frac{1}{2} D_{kij} \big(R_{kj} \nabla_i f - R_{ij} \nabla_k f \big) \\ &= -\frac{n-2}{2} |D|^2. \end{aligned}$$

Finally, integrating this identity over the closed manifold M and substituting it back into our integrated Bach tensor formula, we obtain

$$\begin{aligned}\frac{m(n-2)}{m+n-2}\int_M B_{ij}\nabla_i f\nabla_j f\, dV_g &= -\int_M D_{kij}\nabla_i f\nabla_a\nabla_j f\, dV_g\\ &= -\frac{n-2}{2}\int_M |D|^2\, dV_g.\end{aligned}$$

Dividing both sides by the common factor $(n-2)$, we arrive at the final integral identity

$$\frac{m}{m+n-2}\int_M B_{ij}\nabla_i f\nabla_j f\, dV_g = -\frac{1}{2}\int_M |D|^2\, dV_g. \tag{4.2}$$

This result implies that if the Bach tensor vanishes identically on M, then the D-tensor also vanishes. At a regular point of f, we can choose a local orthonormal frame such that $e_1 = \nabla f/|\nabla f|$, and we denote the components of the Cotton tensor by $C_{ijk} = C(e_i, e_j, e_k)$. By the symmetry properties of the Weyl tensor and the relation $C_{ijk} = -W_{ijkl}\nabla_l f$ from Lemma 3.1.1, it follows that

$$C_{ij1} = -\frac{1}{|\nabla f|}W_{ijkl}\nabla_k f\nabla_l f = \frac{1}{|\nabla f|}W_{ijlk}\nabla_k f\nabla_l f = -C_{ij1},$$

which allows us to conclude that $C_{ij1} = 0$.

Next, let $a, b, c \geqslant 2$ be coordinate indices. Using **Proposition 3.1.2**, we observe that $R_{1a} = 0$ and $R_{1abc} = 0$. By invoking the standard definition of the Weyl curvature tensor, we find

$$\begin{aligned}W_{abc1} &= R_{abc1} - \frac{1}{n-2}\big(R_{ac}g_{1b} + R_{1b}g_{ac} - R_{1a}g_{bc} - R_{bc}g_{1a}\big)\\ &\quad + \frac{R}{(n-1)(n-2)}\big(g_{ac}g_{b1} - g_{1a}g_{bc}\big)\\ &= R_{abc1}.\end{aligned}$$

Since $R_{abc1} = R_{1cba} = 0$, we obtain $W_{abc1} = 0$, which yields $C_{abc} = W_{abc1} = 0$.

Finally, it remains to show that $C_{1ij} = 0$ for all $i, j = 1, 2, \ldots, n$. Indeed, since $D = 0$ and the Bach tensor vanishes, we have

$$\begin{aligned}0 = (n-2)B_{ij} &= |\nabla f|\frac{n-3}{n-2}C_{1ij} - |\nabla f|\frac{1}{m}W_{i1jl}\nabla_l f\\ &= |\nabla f|\frac{n-3}{n-2}C_{1ij} + |\nabla f|\frac{1}{m}W_{1ijl}\nabla_l f\\ &= |\nabla f|\frac{n-3}{n-2}C_{1ij} + |\nabla f|\frac{1}{m}C_{1ij}\\ &= |\nabla f|\left(\frac{n-3}{n-2} + \frac{1}{m}\right)C_{1ij}.\end{aligned}$$

Observe that when $n = 4$, we have $m \neq -(n-2)/(n-3)$. Indeed, for $n = 4$, $-(n-2)/(n-3) = -2$, which is excluded by the hypotheses of the theorem. Consequently, $C_{1ij} = 0$.For $n \geqslant 5$, it is sufficient to prove that $C_{1ab} = 0$ for $a, b \geqslant 2$. First, note that by applying $R_{ab} = \mu\delta_{ab}$ from **Proposition 3.1.2** together with (1.8), we obtain

$$\begin{aligned}
C_{1ab} = -W_{1a1b}|\nabla f|^2 &= \left(-R_{1a1b} + \frac{1}{n-2}\left(R_{11}g_{ab} + R_{ab}\right) - \frac{R}{(n-1)(n-2)}g_{ab}\right)|\nabla f|^2 \\
&= \left(-R_{1a1b} + \frac{1}{n-2}\left(R_{11}g_{ab} + \mu g_{ab}\right) - \frac{R}{(n-1)(n-2)}g_{ab}\right)|\nabla f|^2 \\
&= \left(-R_{1a1b} - \left(\frac{R}{(n-1)(n-2)} - \frac{1}{n-2}(R_{11} + \mu)\right)g_{ab}\right)|\nabla f|^2.
\end{aligned}$$

Since $|\nabla f|$ is constant from previous results, we have

$$[e_a, \nabla f] = [e_a, |\nabla f| e_1] = 0.$$

Furthermore, using $\langle e_1, e_1\rangle = 1$ and $\langle e_1, e_a\rangle = 0$, we obtain $\nabla_1 e_1 = 0$. Indeed, from its decomposition,

$$\nabla_1 e_1 = \sum_{a=2}^{n}\langle \nabla_1 e_1, e_a\rangle e_a + \langle \nabla_1 e_1, e_1\rangle e_1 = \frac{1}{2}e_1(\langle e_1, e_1\rangle)e_1 = 0.$$

Observe that we also have

$$\nabla^{\perp}_a e_b = \langle \nabla_a e_b, e_1\rangle e_1 = -\langle e_b, \nabla_a e_1\rangle e_1 = -h_{ab}e_1.$$

We can now calculate R_{1a1b} directly

$$\begin{aligned}
R_{1a1b} &= \langle \nabla_1\nabla_a e_b - \nabla_a\nabla_1 e_b, e_1\rangle \\
&= \langle \nabla_1(\nabla^{\Sigma}_a e_b + \nabla^{\perp}_a e_b), e_1\rangle - \langle \nabla_a\nabla_1 e_b, e_1\rangle \\
&= \langle \nabla_1(\nabla^{\Sigma}_a e_b - h_{ab}e_1), e_1\rangle - \langle \nabla_a\nabla_1 e_b, e_1\rangle.
\end{aligned}$$

Since $\langle \nabla_1 e_b, e_1\rangle = e_1(\langle e_b, e_1\rangle) - \langle e_b, \nabla_1 e_1\rangle = 0$, it follows that

$$\begin{aligned}
-\langle \nabla_a\nabla_1 e_b, e_1\rangle &= -e_a(\langle \nabla_1 e_b, e_1\rangle) + \langle \nabla_1 e_b, \nabla_a e_1\rangle \\
&= \langle \nabla_b e_1, \nabla_a e_1\rangle \\
&= h_{ac}h_{cb}.
\end{aligned}$$

Using these results, we find that

$$\begin{aligned}
R_{1a1b} &= \langle \nabla_1\nabla^{\Sigma}_a e_b, e_1\rangle + \langle \nabla_1(-h_{ab}e_1), e_1\rangle - \langle \nabla_a\nabla_1 e_b, e_1\rangle \\
&= -\langle \nabla^{\Sigma}_a e_b, \nabla_1 e_1\rangle - \nabla_1(h_{ab}) + h_{ac}h_{cb} \\
&= -\frac{1}{(n-1)}\nabla_1(H)g_{ab} + \frac{H^2}{(n-1)^2}g_{ab}.
\end{aligned}$$

Taking the trace of R_{1a1b} yields

$$R_{11} = -\nabla_1(H) + \frac{H^2}{(n-1)},$$

which allows us to write

$$\begin{aligned} R_{1a1b} &= \frac{1}{(n-1)}\left(-\nabla_1(H) + \frac{H^2}{(n-1)}\right) g_{ab} \\ &= \frac{1}{(n-1)} R_{11} g_{ab}. \end{aligned}$$

Now, observing that

$$\sum_{a=2}^{n} R_{aa} = (n-1)\mu,$$

the scalar curvature decomposes as

$$R = R_{11} + \sum_{a=2}^{n} R_{aa} = R_{11} + (n-1)\mu.$$

Substituting this decomposition into the expression yields

$$\begin{aligned} W_{1a1b} &= -R_{1a1b} - \left(\frac{R_{11} + (n-1)\mu}{(n-1)(n-2)} - \frac{1}{n-2}(R_{11} + \mu)\right) g_{ab} \\ &= -\frac{1}{(n-1)} R_{11} g_{ab} - \left(\frac{R_{11}}{(n-1)(n-2)} - \frac{R_{11}}{n-2}\right) g_{ab} \\ &= -\frac{1}{(n-1)} R_{11} g_{ab} + \frac{1}{n-1} R_{11} g_{ab} = 0. \end{aligned}$$

We conclude that

$$C_{1ab} = W_{1a1b}|\nabla f|^2 = 0 \quad \text{for all } a, b \geqslant 2,$$

which completes the proof.

□

By combining [13, Theorem 1.2], the global classification of warped product Einstein manifolds with harmonic Weyl tensor, and **Theorem 4.1.1**, the geometric constraint $W_{lijk}\nabla_l f \nabla_k f = 0$ implies the corollary below

Corollary 4.1.1. *Let $m \neq 1$ be a positive number. Suppose that (M^n, g, f) with $n \geqslant 4$ is a simply connected quasi-Einstein manifold with $\lambda > 0$ and has a vanishing Bach tensor. Then (M^n, g, f) is either*

(i) *Einstein with a constant function f, or*

(ii) *$g = dt^2 + \psi^2(t) g_L$ and $f = f(t)$, where g_L is Einstein with nonnegative Ricci curvature, and has constant curvature if $n = 4$.*

4.2 Fourth dimensional case

This section is dedicated to examining a specific case directly motivated by **Theorem 4.2.1** and several results found in [13]. To prove this result, we analyze each scenario independently, showing that in every case, the Weyl tensor W_{ijkl} vanishes.

Theorem 4.2.1. *Suppose* (M^4, g, f) *is a compact quasi-Einstein manifold with* $m \neq 0, 1, -2$. *If the Bach tensor B vanishes everywhere, then* Md *is locally conformally flat.*

Proof. Let (M^4, g, f) be a compact 4-quasi-Einstein manifold with $m \neq 0, 1, -2$, and assume the Bach tensor vanishes everywhere on M. Setting $n = 4$, the structural constraint $m \neq 2 - n$ from **Theorem** (4.1.1) reads precisely as $m \neq -2$. Since this value is explicitly excluded by our hypothesis, the conditions of **Theorem** (4.1.1) are satisfied. Consequently, we conclude that M has a harmonic Weyl tensor and satisfies the alignment condition $W_{ijkl}\nabla_l f = 0$.

To prove that M is locally conformally flat, we must show that $W_{ijkl} = 0$ identically. At any regular point of f, we choose a local orthonormal frame $\{e_i\}_{i=1}^4$ where $e_1 = \nabla f/|\nabla f|$, and let $a, b, c, d \in \{2, 3, 4\}$ denote the spatial directions orthogonal to e_1. We analyze and eliminate all possible components of W_{ijkl} via the following four cases

Case 1: Components with at least one index equal to 1.
By **Theorem 4.1.1**, we have

$$W_{ijk1} = \frac{1}{|\nabla f|} W_{ijkl}\nabla^l f = 0,$$

which yields the desired result for this case.

Case 2: Purely spatial components with two repeated pairs (W_{abab}).
For $b \in \{2, 3, 4\}$, expanding the trace $\sum_{i=1}^4 W_{ibib} = 0$ in an orthonormal frame gives $W_{1b1b} + W_{2b2b} + W_{3b3b} + W_{4b4b} = 0$. Using $W_{1b1b} = 0$ (from Case 1) and $W_{bbbb} = 0$ (by anti-symmetry), evaluating for $b = 2, 3, 4$ yields

$$W_{3232} + W_{4242} = 0,$$

$$W_{2323} + W_{4343} = 0,$$

$$W_{2424} + W_{3434} = 0.$$

Since $W_{abab} = W_{baba}$, this system has only the trivial solution

$$W_{2323} = W_{2424} = W_{3434} = 0.$$

Case 3: Purely spatial components with one repeated pair (W_{aacb} or W_{abac}).
Next, we examine components where one pair of indices is repeated, such as W_{2233}, W_{3324}, or W_{abac} where $b \neq c$. By the fundamental pairwise skew-symmetry of the Weyl tensor on its first two slots, any component where the first two indices are identical is trivially zero

$$W_{aacb} = -W_{aacb} = 0 \implies W_{2233} = 0.$$

For components of the form W_{abac} where the repetition occurs across different pairs, we apply the trace-free condition once more ($\sum_{i=1}^{4} W_{ibic} = 0$ for $b \neq c$)

$$W_{1b1c} + W_{2b2c} + W_{3b3c} + W_{4b4c} = 0.$$

Since $W_{1b1c} = 0$ from Case 1, and the terms where $i = b$ or $i = c$ vanish by internal anti-symmetry (e.g., $W_{bbbc} = 0$), the remaining single term must vanish. For example, selecting $b = 2$ and $c = 4$ forces

$$W_{3234} = 0 \implies W_{3324} = 0.$$

As a result, all components with one repeated pair are zero.

Case 4: Purely spatial components with distinct indices.
Since the spatial index set $\{2, 3, 4\}$ has cardinality 3, it is impossible to choose four distinct indices a, b, c, d. Hence, no such components exist, and this case holds vacuously.

Combining Cases 1 through 4, the Weyl tensor vanishes on the set of regular points of f. By continuity, $W \equiv 0$ everywhere on M^4, which shows that (M^4, g) is locally conformally flat.

□

In [13], He, Petersen, and Wylie studied warped product Einstein manifolds with a non-empty boundary ∂M. Following their setup, let $u = e^{-f/m}$ in the interior of M and $u = 0$ on ∂M. As noted in their work, both **Theorem**(4.1.1) and **Theorem**(4.2.1) extend naturally to the case where M has a non-empty boundary.

To see this, for any $\varepsilon > 0$, one considers the regular sublevel sets $M_\varepsilon = \{x \in M : u(x) \geqslant \varepsilon\}$. It is sufficient to show that $D = 0$ on M_ε, since sending $\varepsilon \to 0^+$ yields $D = 0$ on the entire manifold M. Indeed, the boundary integral in (4.1) vanishes:

$$\int_{\partial M_\varepsilon} D(\nu, \nabla f, \nabla f)\, d\mathrm{vol} = 0,$$

because the unit outward normal vector ν along ∂M_ε is parallel to ∇f. Consequently, the integral identity (4.2) holds on M_ε, from which it follows that $D = 0$ on M_ε.

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